After the Invitation to Bessel Functions

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To my Family

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Preface

The Bessel functions, because of their wide use and applications, have various generalizations. Among them are for example the two-, three- and four-index analogues of the Bessel functions, as well as the hyper-Bessel functions.

The topics of this book include enumerable families of special functions, as the above listed, namely Bessel functions and their generalizations with two, three and four indices, briefly called Bessel type functions in the book. Various properties, asymptotic formulae, integration operators and some of their applications, zeros of entire functions represented by integrals, involving Bessel functions, convergence of series in such families of special functions are studied. Results, analogical to the classical ones for the power series are obtained, and the conclusion is that each of considered series has a behaviour like a power series.

The present book has to be considered as a monograph, since it treats a specialized topic as well as since it is based mostly on the Authors own works. The book consists of introduction, six chapters and bibliography.

The introduction gives a brief historical overview of the subject matter of this book. It traces the arising of the special functions, listed above, as well as the problems concerning them, giving the motivation for further studies and applications.

In Chapter 1 we include some preliminary results on the Bessel functions, related with them Bessel-Clifford functions and their generalizations, quoting their definitions. They serve to make the book self-contained.

Chapter 2 has a preparatory nature. Some basic and well-known in the literature integral representations and asymptotic formulae for the Bessel functions are given. Besides, asymptotic formulae are proposed for the generalized Bessel functions that are necessary for use in Chapter 3 and 4. For illustrations, $3 - D$ representations are given, using CAS 'Maple' and taking different values for the parameters.

Chapter 3 considers series in Bessel functions in the complex plane and studies their convergence, giving results analogical to the classical ones for the widely used power series.

In Chapter 4 we consider series by means of Bessel type functions, essentially using the asymptotic formulae for them, obtained in Chapter 2. Studying them, we obtain various results connected to their geometry of convergence, such as Cauchy-Hadamard, Abel and Tauber type theorems.

Chapter 5 is devoted to the asymptotic behaviour of zeros of finite Hankel transforms. The investigations are based on a Hurwitz theorem, who consider appropriate meromorphic functions instead of the entire functions.

Chapter 6 deals with a practical problem, using the finite integral Hankel transformation for solving. A mathematical model of non-stationary heat convection of power plants with liquid-rocket engines of non-piloted flying devices working at rate of strong throttling of the pulling power is represented there. The process is modeled for using the finite integral Hankel transformation for a hollow axially symmetrical cylinder with third kind boundary conditions on the inner and outer combustion chamber wall. Using the graphic of Maple 13 for Windows, it becomes possible to prognosticate the intensity of the overheating of the chamber and the nozzle at starting the engine.

The bibliography consists of 68 items, published up to 2018. However, it does not pretend to be considered as a complete list and interested reader may find additional references in the monographs and surveys mentioned in Introduction.

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Introduction

Bessel functions appeared in solving of concrete problems of mechanics and astronomy. They had approved themselves as ones of the most frequently used special functions in mathematical analysis and its applications in physics, mechanics and engineering sciences.

They represent the solutions of many problems in the mentioned areas and give a opportunity for their detailed study, based on the broad and comprehensive information for the Bessel functions, gathered during almost two centuries.

As examples can be mentioned the axial-symmetric problems, whose analytic treating leads to linear partial differential equations, containing the Laplace operator. Their solutions, obtained by the Fourier method of separating of the variables, in cylindric coordinates, are expressed by Bessel functions. Because of that, the Bessel functions, as solutions of the Bessel differential equation, are called also cylindric functions.

In the frames of the classical Complex analysis and Ordinary differential equations theory, the Bessel functions are considered as solutions of a linear differential equation of second order with a regular singularity at the origin and irregular singularity at the infinity. This makes the analytic nature of the Bessel functions excessively clear, which, in general, are multi-valued analytic functions with 'power' and/or 'logarithmic' singularity at the origin, respectively at the infinity. Considering the Bessel functions as a special class of analytical functions completely reveals their nature. In particular, this refers to their asymptotic expansions, as well as various integral representations. As a result of the long-time investigations, a great amount of results has been created which is widely used in Mathematical analysis and its applications.

The theory of Bessel functions, details for them and their properties and applications can be found in the monographs by Watson [66], Erdélyi et al. (ed-s) [7], Samko, Kilbas and Marichev [56], Whittaker and Watson [67], Dzrbashjan [6], Prudnikov [47], contemporary monograph by Rusev [55], and so on.

As it is well known, the studying of the properties of the complex-valued functions, which are holomorphic in a complex domain is often based on the possibility of their representations by series in concrete countable systems of functions, holomorphic in the considered domain. For circular domains, most frequently the Taylor systems are attracted, that leads to the power series representations. In a 'right', respectively 'left' plane, the Dirichlet type series are used. In domains of more general nature are engaged series in Faber polynomials. Series in classical orthogonal polynomials are also used.

The study of the series in the system $\{J_n(z)\}_{n=0}^{\infty}$ of Bessel functions of first kind with nonnegative integer indices can be traced back to the nineteenth century by Carl Neumann. He presented the Cauchy kernel by bilinear series in these functions and the Neumann 'polynomials'. Using the last series, it has been proved that the system $\{J_n(z)\}_{n=0}^{\infty}$ is a basis in the space of complex functions, holomorphic in the open disk $D(0; R)$ $(0 < R \leq \infty)$, centered at the origin and with a radius R. The basicity problem of countable systems of Bessel functions has been proved later by other authors, for example Friedrich Schäfke [59], [60], [61], M. Lehua [23], Paneva-Konovska [35], and so on. Nowadays, Bessel's functions have various useful generalizations obtained by adding additional indices including two-, three- and four-index Bessel-Wright functions.

Along with the Bessel functions systems, and the results referring to the convergent series in them, suitable enumerable families of the Bessel type functions are also specified and considered series in these functions in the complex plane $\mathbb{C}$. Such a kind of results are provoked by the fact that the solutions of some differential and integral equations can be written in terms of series (or integrals, or series of integrals) of special functions, see for example in Kiryakova [16], Sandev, Tomovski and Dubbeldam [58], Sandev, Deng and Xu[57], Herzallah and Baleanu [9], for other applications see also [45], [46], [65]. In studying their convergence, asymptotic formulae have been obtained, referring to the 'large' values of indices.

They are further used for determining the domains of convergence, i. e. where the series converges and where it diverges, where the convergence is uniform and where it is not. The boundary behaviour of the series is also investigated, proving theorems analogical to the ones for the power series, namely the classical Abel and Tauber theorems.

Regarding the results related to the convergence of considered series in various families, obtained in this monograph, we can briefly summarize that they are completely analogical to the ones, connected with the widely used power series.

Closely related with Bessel functions is the Hankel transform, which is also known as the Fourier–Bessel transform. Studying a class of entire functions of an exponential kind, using the ideas of Jensen and Riemann, the distribution and the asymptotic of their zeros are found. Distribution of zeros of Hankel transform, involving Bessel functions in the kernel, is also discussed and found.

As application, a mathematical model of non-stationary heat convection of power plants with liquid-rocket engines of non-piloted flying devices working at rate of strong throttling of the pulling power is represented. The process is modeled for using the finite integral Hankel transform for a hollow axially symmetrical cylinder with third kind boundary conditions on the inner and outer combustion chamber wall. For more general miscellaneous useful properties and applications of a number operators, see also the recent survey paper [1] by Bazhlekova, as well as Kiryakova [16] and [18].

1 Bessel functions and Related to them

1.1 Bessel Functions

The solving of a set of problems in the Mechanics and Mathematical Physics is closely related to the Bessel differential equation

$$z^2\frac{d^2w}{dz^2} + z\frac{dw}{dz} + (z^2 - \nu^2)w = 0. \tag{1.1.1}$$

The function $J_\nu(z)$, defined by the equality

$$J_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{2k+\nu}}{k!\,\Gamma(k+\nu+1)}, \quad z \in \mathbb{C} \setminus (-\infty, 0], \tag{1.1.2}$$

is a solution of this equation in the domain $\mathbb{C} \setminus (-\infty, 0]$ (see e.g. [66], 3.1 (8)). It is called *Bessel function of first kind with an index* ν. The function $J_{-\nu}(z)$ is also a solution of the above equation. The Bessel functions of first kind with an integer index are called also *Bessel coefficients*. They are holomorphic in the whole complex plane.

First, let us consider the case when the parameter ν is not integer. Then the linear combinations (see [7], 7.2 (4)–(6)):

$$Y_\nu(z) = \frac{J_\nu(z)\cos(\nu\pi) - J_{-\nu}(z)}{\sin(\nu\pi)}, \quad z \in \mathbb{C} \setminus (-\infty, 0], \tag{1.1.3}$$

$$H_\nu^{(1)}(z) = J_\nu(z) + i\,Y_\nu(z), \quad z \in \mathbb{C} \setminus (-\infty, 0], \tag{1.1.4}$$

$$H_\nu^{(2)}(z) = J_\nu(z) - i\,Y_\nu(z), \quad z \in \mathbb{C} \setminus (-\infty, 0]. \tag{1.1.5}$$

are also solutions of the differential equation (1.1.1). $Y_\nu(z)$ are called *Bessel functions of second kind*, and $H_\nu^{(1)}(z)$, $H_\nu^{(2)}(z)$ – *Bessel functions of third kind*, or also *first and second Hankel functions.*

If ν is an integer, then the left hand side of the equalities (1.1.3)–(1.1.5) are not defined. But their limits, when $\nu \to n$ (n is an integer), exist and they can be used to define the Bessel functions of second and third kind with an integer index. In particular, we have $Y_n(z) = \lim\limits_{\nu \to n} Y_\nu(z)$ ([7], 7.2 (28)), i.e.

$$Y_n(z) = \frac{1}{\pi}\left[\frac{\partial J_\nu(z)}{\partial \nu} - (-1)^n \frac{\partial J_{-\nu}(z)}{\partial \nu}\right]_{\nu=n}, \quad z \in \mathbb{C} \setminus (-\infty, 0], \tag{1.1.6}$$

The functions $J_\nu(z)$ and $J_{-\nu}(z)$ form a fundamental system of solutions of the differential equation (1.1.1) iff ν is not integer ([66], 3.12), whereas $J_\nu(z)$ and $Y_\nu(z)$ always form a fundamental system of solutions of this equation ([66], 3.63). One of the primer motives for introducing the functions $Y_\nu(z)$ is the necessity of second solution that is linearly independent of $J_\nu(z)$, when $\nu = n$ is a nonnegative integer.

1.2 Modified Bessel functions

The solving of some problems in the Mathematical Physics often is related to the differential equation

$$z^2\frac{d^2w}{dz^2} + z\frac{dw}{dz} - (z^2+\nu^2)w = 0, \qquad (1.2.1)$$

that differs from the Bessel equation by the coefficient of w and it can be obtained by (1.1.1) replacing iz instead of z.

The system $\{J_\nu(iz), J_{-\nu}(iz)\}$, as well as $\{J_\nu(iz), Y_\nu(iz)\}$, are fundamental systems of solutions of the equation (1.2.1) in the domain $z \in \mathbb{C} \setminus (-\infty, 0]$, but more frequently are used the functions ([66], 3.7 (2), [7], 7.2 (12))

$$I_\nu(z) = \sum_{k=0}^{\infty} \frac{(z/2)^{2k+\nu}}{k!\,\Gamma(k+\nu+1)}, \quad z \in \mathbb{C} \setminus (-\infty, 0] \qquad (1.2.2)$$

and $I_{-\nu}(z)$. They are called *modified Bessel functions of first kind*. The Bessel and modified Bessel functions of the first kind are related with simple dependence in corresponding domains of the complex plane, namely:

$$I_\nu(z) = \exp(-i\nu\pi/2) J_\nu(iz), \quad -\pi < \arg z < \pi/2,$$

$$I_\nu(z) = \exp(i\nu\pi/2) J_\nu(-iz), \quad -\pi/2 < \arg z < \pi.$$

The function ([7], 7.2 (13), (36))

$$K_\nu(z) = \frac{\pi(I_{-\nu}(z) - I_\nu(z))}{2\sin(\nu\pi)}, \quad \nu \notin \mathbb{Z},\ z \in \mathbb{C} \setminus (-\infty, 0],$$

$$K_n(z) = \frac{(-1)^n}{2}\left[\frac{\partial I_{-\nu}(z)}{\partial \nu} - \frac{\partial I_\nu(z)}{\partial \nu}\right]_{\nu=n}, \ n \in \mathbb{Z},\ z \in \mathbb{C} \setminus (-\infty, 0],$$

is also a solution of the equation (1.2.1). It is called *modified Bessel function of third kind*, although the modern definition is given by MacDonald ([7], 7.2.2).

The functions with an index of the kind $n+1/2$ ($n = 0, \pm 1, \dots$), called *Bessel functions with a half-integer index* or also *spherical Bessel functions*, form an interesting class of functions. They can be expressed as rational functions of $\sqrt{z}$, $\cos z$, $\sin z$ and $\exp z$. In particular, the modified Bessel functions of third kind with half-integer indices satisfy the relations ([7], 7.2 (40), (42), 7.3 (16)):

$$K_{n+1/2}(z) = \sqrt{\frac{\pi}{2z}} \exp(-z) \sum_{k=0}^{n} \frac{(2z)^{-k}\Gamma(n+k+1)}{k!\,\Gamma(n-k+1)}, \ |\arg z| < \pi, \tag{1.2.3}$$

$$K_{1/2}(z) = \sqrt{\frac{\pi}{2z}} \exp(-z), \quad |\arg z| < \pi. \tag{1.2.4}$$

1.3 Bessel–Clifford functions

Closely related to the Bessel functions $J_\nu(z)$, are the so-called Bessel–Clifford functions $C_\nu(z)$. These functions are entire functions of z. and they have the following representations in the complex plane:

$$C_\nu(z) = z^{-\nu/2} J_\nu(2\sqrt{z}) = \sum_{k=0}^{\infty} \frac{(-1)^k (z)^k}{k!\, \Gamma(\nu + k + 1)}\,, \quad \nu \in \mathbb{C} \tag{1.3.1}$$

1.4 Generalizations of Bessel functions of first kind

The Bessel functions and their various generalizations, originating from concrete problems in mechanics and astronomy, have proved themselves as some of the most frequently used special functions in mathematical analysis and its applications in physics, mechanics and engineering.

Generalizations of the Bessel functions (more precisely, of the Bessel–Clifford functions) involving one more additional index μ have been introduced by Wright [68] and called Bessel–Wright functions or also misnamed in the literature as Bessel–Maitland functions (after Sir Edward Maitland Wright), namely:

$$J_\nu^\mu(z) = \sum_{k=0}^{\infty} \frac{(-z)^k}{k!\, \Gamma(\nu + \mu k + 1)}\,, \quad \mu > -1, \tag{1.4.1}$$

for details, see Marichev [26, p.109]; [12, p.336], etc. Initially, Wright defined (1.4.1) only for $\mu > 0$, and on a later stage extended its definition to $\mu > -1$ (see for example [12], [15]).

More general are the three- and four-index generalizations of the Bessel function J_ν, namely generalized Bessel–Maitland (or Wright) functions introduced by Pathak [42] (for details see also [14]):

$$J^{\mu}_{\nu,\lambda}(z) = (z/2)^{\nu+2\lambda} \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{2k}}{\Gamma(\lambda+k+1)\Gamma(\nu+k\mu+\lambda+1)}, \qquad z \in \mathbb{C} \setminus (-\infty, 0]\,; \quad \mu > 0, \quad \nu, \lambda \in \mathbb{C}, \tag{1.4.2}$$

and the generalized Lommel–Wright functions, introduced by de Oteiza, Kalla and Conde (for details and results related to fractional calculus see [14] and also [44])

$$J^{\mu,m}_{\nu,\lambda}(z) = (z/2)^{\nu+2\lambda} \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{2k}}{(\Gamma(\lambda+k+1))^m\, \Gamma(\nu+k\mu+\lambda+1)} \qquad z \in \mathbb{C} \setminus (-\infty, 0]\,; \quad \mu > 0, \quad m \in \mathbb{N}, \quad \nu, \lambda \in \mathbb{C}. \tag{1.4.3}$$

One more interesting generalization is the *hyper-Bessel function* $J^{(m)}_{\nu_1,\ldots,\nu_m}$, defined by the formula

$$J^{(m)}_{\nu_1,\ldots,\nu_m}(z) = \frac{\left(\frac{z}{m+1}\right)^{\nu_1+\cdots+\nu_m}}{\Gamma(\nu_1+1)\ldots\Gamma(\nu_m+1)}\; j^{(m)}_{\nu_1,\ldots,\nu_m}(z)\,, \tag{1.4.4}$$

where $z,\ \nu_i \in \mathbb{C}$, $Re(\nu_i+1) > 0$ $(i = 1, \ldots, m)$, and

$$j^{(m)}_{\nu_1,\ldots,\nu_m}(z) = \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{z}{m+1}\right)^{k(m+1)}}{(\nu_1+1)_k \ldots (\nu_m+1)_k}\, \frac{1}{k!}, \qquad |z| < \infty. \tag{1.4.5}$$

In view of (1.4.5), the hyper-Bessel functions can be written in the form:

$$J^{(m)}_{\nu_1,\dots,\nu_m}(z) = \left(\frac{z}{m+1}\right)^{\sum\limits_{i=1}^{m}\nu_i} \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{z}{m+1}\right)^{k(m+1)}}{\Gamma(k+\nu_1+1)\dots\Gamma(k+\nu_m+1)} \frac{1}{k!}. \tag{1.4.6}$$

In 1953, this function was introduced by Delerue [2] as a natural generalization of order m with vector indices $\nu = (\nu_1, \nu_2, \dots, \nu_m)$ (or with multi-index $(\nu_1, \dots, \nu_m)$) of the Bessel function of the first type J_ν. Later, this function was also studied by other authors, for example by Marichev [26], Dimovski [3], Kljuchantcev [20], [21], Dimovski and Kiryakova [4]–[5], Kiryakova [12], [13], [17], Paneva-Konovska, etc.

The hyper-Bessel functions of Delerue are closely related to the *hyper-Bessel differential operators* of arbitrary order $m > 1$, *introduced by Dimovski* [3]. These are singular linear differential operators that appear very often in problems of mathematical physics as a generalization of the 2nd order Bessel operator that can be represented in the alternative forms

$$B = z^{\alpha_0}\frac{d}{dz}z^{\alpha_1}\dots\frac{d}{dz}z^{\alpha_m} = z^{-\beta}\prod_{k=1}^{m}\left(z\frac{d}{dz}+\beta\gamma_k\right) \tag{1.4.7}$$

$$= z^{-\beta}\left(z^m\frac{d^m}{dz^m} + a_1 z^{m-1}\frac{d^{m-1}}{dz^{m-1}} + \dots + a_{m-1}z\frac{d}{dz} + a_m\right),$$

$0 < z < \infty$, with sets of $(m+1)$ parameters $\{\alpha_0, \alpha_1, ..., \alpha_m\}$, or $\{\beta > 0,\ \gamma_k \text{ real},\ k = 1, ...m\}$, or $\{\beta > 0,\ a_1, ..., a_m\}$. For details, see also

Dimovski and Kiryakova [4], [5], and Kiryakova [12, Ch.3]. Indeed, as shown in Th. 3.4.3 and Cor. 3.4.4 in Kiryakova [12], the fundamental system of solutions of the m-th order *hyper-Bessel differential equation* $By(z) = \lambda\, y(z),\ \lambda \neq 0$ consist of the set of hyper-Bessel functions

$$J^{(m-1)}_{1+\gamma_1-\gamma_k,\ldots,*,\ldots,1+\gamma_m-\gamma_k}\left[(-\lambda)^{1/m}(m/\beta)\, z^{\beta/m}\right],\quad k=1,\ldots,m,$$

under assumption of formal arrangement of the γ-parameters as $\gamma_1 < \gamma_2 < \ldots < \gamma_m < \gamma_1 + 1$ and where $*$ means to omit the k-th term in the indices. And then, the solutions of hyper-Bessel ODEs $By(z) = \lambda\, y(z) + f(z)$ can be given explicitly in terms of hyper-Bessel functions, series in them, or series in integrals of them ([12]).

Evidently, the hyper-Bessel functions (1.4.6) are natural generalizations of the *Bessel function* of the first kind (with $m+1=2$, $m=1$), i.e.

$$J^{(1)}_{\nu}(z) = J_\nu(z) = \left(\frac{z}{2}\right)^{\nu} \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{z}{2}\right)^{2k}}{\Gamma(k+\nu+1)}\, \frac{1}{k!}, \tag{1.4.8}$$

as well as of the so-called *Bessel–Clifford functions of 3rd order* ($m+1=3$): $C_{\nu,\mu}(z)$, depending on two ($m=2$) indices and modifying the hyper-Bessel functions $J^{(2)}_{\nu,\mu}(z)$, namely

$$C_{\nu,\mu}(z) = z^{-\frac{\mu+\nu}{3}}\ J^{(2)}_{\nu,\mu}(3\sqrt[3]{z}) = \sum_{k=0}^{\infty} \frac{(-1)^k\,(z)^k}{\Gamma(k+\mu+1)\Gamma(k+\nu+1)}\, \frac{1}{k!} \tag{1.4.9}$$

see details in [19]. Other interesting special cases of the hyper-Bessel functions of arbitrary order but with specific choice of indices are the so-called *trigonometric functions of order* m, including the *generalized* $\cos_m$- and the *generalized* $\sin_{m,k}$*-functions* ($k = 1, ..., m-1$), studied for example by Kljuchantzev [20], [21], Dimovski and Kiryakova [4], [5], Kiryakova [12], [13], etc. They appear as solutions of well-known classical case with particular hyper-Bessel operator $B = \left(\frac{d}{dz}\right)^m$. For example, the solution of the Cauchy problem

$$y^{(m)}(z) = -y(z), \quad y(0) = 1, y^{'}(0) = ... = y^{(m-1)}(0) = 0,$$

is given by:

$$y(z) = \cos_m(z) = \sum_{k=0}^{\infty} \frac{(-1)^k\, z^{mk}}{(mk)!} = j^{(m-1)}_{\nu_1,\ldots,\nu_{m-1}}(z) \ \text{ with } \ \nu_k := \frac{k}{m} - 1.$$

At last, in particular (see e.g. [26, p.110]; [12, p. 352–353]):

$$\begin{aligned} J^1_\nu(z) = C_\nu(z), \quad J^1_{\nu,0}(z) = J^{(1)}_\nu(z) = J_\nu(z), \quad J^\mu_{\nu,0}(z) = (z/2)^\nu J^\mu_\nu(z^2/4), \\ J^{\mu,1}_{\nu,\lambda}(z) = J^\mu_{\nu,\lambda}(z), \quad J^1_{\nu,\lambda}(z) = \frac{2^{2-2\lambda-\nu}}{\Gamma(\lambda)\Gamma(\lambda+\nu)}\, s_{2\lambda+\nu-1,\nu}(z), \\ \mathbf{H}_\nu(z) = \frac{2^{1-\nu}}{\sqrt{\pi}\Gamma(\nu+1/2)}\, s_{\nu,\nu}(z) = J^1_{\nu,1/2}(z), \end{aligned} \tag{1.4.10}$$

where $s_{\alpha,\nu}(z)$ and $\mathbf{H}_\nu(z)$ denote respectively the Lommel and Struve functions [7, 7.5.5, (69), (84), Vol. 2].

In what follows, for the sake of brevity and in order to make the further presentation clearer, we briefly call the generalizations of the Bessel functions *Bessel type functions*.

2 Integral Representations and Asymptotic Formulae

2.1 Preliminary results

One of the most often used representations of the Bessel functions of the first kind is the Poisson integral representation ([7], 7.12 (7)):

$$J_\nu(z) = \frac{z^\nu}{\sqrt{\pi}2^\nu\Gamma(\nu+1/2)}\int_{-1}^{1}(1-t^2)^{\nu-1/2}\exp(izt)dt \qquad (2.1.1)$$

that holds when $Re\nu > -1/2$. It served for an origin of many important investigations in the area of Bessel's functions.

The modified Bessel functions of the third kind have the representations ([7], 7.12 (27)):

$$K_\nu(z) = \frac{(2z)^\nu\Gamma(\nu+1/2)}{\sqrt{\pi}}\int_{0}^{\infty}(z^2+t^2)^{-\nu-1/2}\cos t\, dt \qquad (2.1.2)$$

that hold when $Re\nu > -1/2$ and $|\arg z| < \pi/2$, i.e., when z lies in the right half-plane $Rez > 0$ and $Re\nu > -1/2$.

Depending on that, whether the index ν or the argument z grow infinitely, different asymptotic formulae are known for the Bessel functions. So, for example, the Bessel coefficients $J_n(z)$ have the following representations (see e.g. [67, §17.81]):

$$J_n(z) = \left(\frac{z}{2}\right)^n (1 + \theta_n(z)) \frac{1}{n!}, \quad \theta_n(z) \to 0 \text{ as } n \to \infty \tag{2.1.3}$$

in the whole complex plane. The functions $\theta_n(z)$ are holomorphic for $z \in \mathbb{C}$ and moreover $\lim\limits_{n\to\infty} \theta_n(z) = 0$ uniformly on each compact subsets of the plane $\mathbb{C}$.

Note that for $\mu = 1$ one can obtain corresponding asymptotics for the Lommel functions, as well.

The Bessel functions have the following representation when $z \to \infty$ ([7], 7.13 (3)):

$$J_\nu(z) = \sqrt{\frac{2}{\pi z}} \left(\cos(z - \lambda_\nu) - \sin(z - \lambda_\nu)\, O\left(\frac{1}{|z|}\right)\right), \tag{2.1.4}$$

with $\lambda_\nu = \dfrac{\nu\pi}{2} + \dfrac{\pi}{4}$, for $\arg|z| \le \pi - \delta$ and arbitrary $0 < \delta < \pi$.

2.2 An upper estimate

Considering explicitly $\theta_n(z)$, we can make the result from (2.1.3), sharper.

Theorem 2.2.1. *Let $K \subset \mathbb{C}$ be a nonempty compact set. Then there exists a constant $C = C(K)$, $0 < C < \infty$, such that for each $n \in \mathbb{N}_0$ and each $z \in K$, the following inequality holds*

$$|\theta_n(z)| \leq C/(n+1). \tag{2.2.1}$$

Proof. First, let $z \in \mathbb{C}$. Due to (1.1.2) and (2.1.3), we can write

$$\theta_n(z) = \frac{1}{n+1} \sum_{k=1}^{\infty} \frac{(-1)^k (n+1)!}{k!(n+k)!} \left(\frac{z}{2}\right)^{2k}.$$

Denoting $u_k(z) = \dfrac{(-1)^k (n+1)!}{k!(n+k)!} \left(\dfrac{z}{2}\right)^{2k}$, we obtain the estimate

$$|u_k(z)| \leq \frac{1}{k!} \left|\frac{z}{2}\right|^{2k} \tag{2.2.2}$$

for the absolute value of $u_k(z)$. The series

$$\sum_{k=1}^{\infty} \frac{1}{k!} \left|\frac{z}{2}\right|^{2k} \tag{2.2.3}$$

converges for each $z \in \mathbb{C}$ and its sum is $\exp(|z|^2/4) - 1$. This shows that

$$|\theta_n(z)| \leq \frac{1}{n+1} \left(\exp\left(|z|^2/4\right) - 1\right) \tag{2.2.4}$$

on the whole complex plane.

Then, the estimate (2.2.1) follows immediately from (2.2.4), for all the values $z \in K$. $\square$

Further, the following remarks can be written.

Remark 2.2.1. *The uniform convergence of* $\theta_n(z)$ *on the compact subsets of* $\mathbb{C}$ *follows from* (2.2.1) *as well.*

Remark 2.2.2. *According to the asymptotic formula* (2.1.3), *it follows that there exists a natural number* N_0 *such that the functions* $J_n(z)$ *have no zeros for* $n > N_0$, *except for the origin.*

Remark 2.2.3. *Note that each of the functions* $J_n(z)$ $(n \in \mathbb{N}_0)$, *being an entire function, not identically zero, has no more than finite number of zeros in the closed and bounded set* $|z| \leq R$ *([27], vol.1, ch. 3, §6, 6.1, p.305). Moreover, because of Remark 2.2.2, no more than finite number of these functions have some zeros, except for the origin.*

2.3 Asymptotic formulae with respect to index ν

In this Section we propose some asymptotic formulae with respect to the index for the generalized Bessel functions and generalizations, i. e. in the case when indices grow to infinity. These formulas are used in studying the properties of series of Bessel–Maitland functions, for example, in the proofs of Cauchy–Hadamard, Abel and Tauber type theorems for series of such functions, see Paneva-Konovska [34], [40].

Our results are natural generalizations of the known asymptotic formulae (2.1.3) for the Bessel functions J_n when the index n is a nonnegative integer with $n \to \infty$

Further we consider the functions (1.4.1) and (1.4.2) for $\mu > 0$. In this case we prove some asymptotic formulae for "large" values of index ν.

Theorem 2.3.1. *Let* $\mu > 0$, $|arg\nu| < \pi$. *Then for the Bessel–Maitland (Wright) functions the following asymptotic formula*

$$J_\nu^\mu(z) = \frac{1}{\Gamma(\nu+1)}(1 + \theta_\nu^\mu(z)), \quad z \in \mathbb{C}, \tag{2.3.1}$$

$$\theta_\nu^\mu(z) \to 0 \ \ as \ \ \mathcal{R}e\, \nu \to \infty$$

is valid. The functions $\theta_\nu^\mu(z)$ *are holomorphic for* $z \in \mathbb{C}$. *The convergence is uniform on the compact subsets of the complex plane* $\mathbb{C}$.

Proof. Let us represent the functions $J_\nu^\mu(z)$ in the form

$$J_\nu^\mu(z) = \frac{1}{\Gamma(\nu+1)}\left(1 + \sum_{k=1}^{\infty} \frac{\Gamma(\nu+1)(-z)^k}{k!\,\Gamma(\nu+\mu k+1)}\right)$$

and for brevity's sake, denote

$$\theta_\nu^\mu(z) = \sum_{k=1}^{\infty} \frac{\Gamma(\nu+1)(-z)^k}{k!\,\Gamma(\nu+\mu k+1)}, \quad w_k(\nu,\mu) = \frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu+\mu k+1)}\,.$$

Then we have:

$$\theta_\nu^\mu(z) = \frac{\Gamma(\nu+1)}{\Gamma(\nu+\mu+1)} \sum_{k=1}^{\infty} \frac{w_k(\nu,\mu)}{k!}\,(-z)^k. \tag{2.3.2}$$

Using the analogues of the Stirling formula (see [7], p.62,(5))

$$\Gamma(z+1) \sim \sqrt{2\pi z}\, z^z \exp(-z), \quad \Gamma(z+\alpha) \sim z^\alpha\, \Gamma(z), \quad \mathcal{R}e\, z \to \infty,$$

we obtain for large values of ν:

$$\lim_{\mathcal{R}e\,(\nu)\to\infty} (\nu^\mu\ w_k(\nu,\mu)) = \begin{cases} 1 & \text{for } k=2 \\ 0 & \text{for } k=3,4,5,\dots \end{cases},$$

and as $\lim\limits_{\mathcal{R}e\,(\nu)\to\infty} (\nu^\mu) = \infty$, then $\lim\limits_{\mathcal{R}e\,(\nu)\to\infty} (w_k(\nu,\mu)) = 0$ for $k = 2,3,4,\dots$. Therefore, there exists a natural number N_0 such that for all ν with $\mathcal{R}e\,(\nu) > N_0$ the inequalities $|w_k(\nu,\mu)| < 1$ hold. Now, let $\mathcal{R}e\,(\nu) > N_0$. We have such estimate $|\frac{w_k(\nu,\mu)}{k!}\ (-z)^k| < \frac{|z|^k}{k!}$ for the module of the common term in the series of the right hand side of (2.3.2). On the other hand, the series $\sum\limits_{k=1}^{\infty} \frac{|z|^k}{k!}$ is convergent in $\mathbb{C}$ and therefore the power series $\sum\limits_{k=1}^{\infty} \frac{(-z)^k}{k!}$ is absolutely convergent on the whole complex plane and uniformly convergent on the compact subsets of $\mathbb{C}$. Therefore, the series in (2.3.2) is also absolutely convergent in the whole plane and uniformly convergent on its compact subsets. The fact that $\theta_\nu^\mu(z)$ are holomorphic functions follows from the convergence of the series in (2.3.2). Moreover, using once again the Stirling formula, we obtain the equality $\lim\limits_{\mathcal{R}e\,(\nu)\to\infty} \left(\frac{\Gamma(\nu+1)}{\Gamma(\nu+\mu+1)}\right) = 0$, which proves the theorem. $\square$

Remark 2.3.1. *Just mention that replacing $\mu = 1$ in the Bessel–Maitland function gives the corresponding asymptotic formula for the Bessel Clifford function* (1.3.1), *namely:*

$$C_\nu(z) = \frac{1}{\Gamma(\nu+1)}(1+\theta_\nu^1(z)), \quad z \in \mathbb{C},$$

$$\theta_\nu^1(z) \to 0 \;\; as \;\; \mathcal{R}e\,\nu \to \infty,$$

is valid. The functions $\theta_\nu^1(z)$ are holomorphic for $z \in \mathbb{C}$. The convergence is uniform on the compact subsets of the complex plane $\mathbb{C}$.

Consider now the generalized Bessel–Maitland (Wright) functions (1.4.2) for indices of the form $\nu = n - 2\lambda,\;\; n = 0, 1, 2, ...,$

$$J^\mu_{n-2\lambda,\lambda}(z) = (z/2)^n \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{2k}}{\Gamma(\lambda+k+1)\Gamma(n-\lambda+k\mu+1)}\,. \qquad (2.3.3)$$

It is not difficult to see that these are entire functions of z.

Theorem 2.3.2. *Let $\mu > 0$. Then for the generalized Bessel–Maitland (Wright) functions* (2.3.3) *the following asymptotic formula*

$$J^\mu_{n-2\lambda,\lambda}(z) = \frac{(z/2)^n}{\Gamma(\lambda+1)\Gamma(n-\lambda+1)}(1+\theta^\mu_{n-2\lambda,\lambda}(z)), \quad z \in \mathbb{C}\,, \qquad (2.3.4)$$

$$\theta^\mu_{n-2\lambda,\lambda}(z) \to 0 \;\; as \;\; n \to \infty \quad (n \in \mathbb{N}),$$

holds. The functions $\theta^\mu_{n-2\lambda,\lambda}(z)$ are holomorphic functions of z in $\mathbb{C}$. The convergence is uniform on the compact subsets of the complex plane $\mathbb{C}$.

Proof. To prove this theorem, we follow the same idea as in the proof of Theorem 2.3.1. First, for the sake of brevity, denote

$$v_k(n;\lambda,\mu) = \frac{1}{\Gamma(\lambda+k+1)\,\Gamma(n-\lambda+\mu k+1)}.$$

Then (2.3.3) gets the form

$$J^{\mu}_{n-2\lambda,\lambda}(z) = (z/2)^n \sum_{k=0}^{\infty} (-1)^k\, v_k(n;\lambda,\mu)\,(z/2)^{2k}, \quad z \in \mathbb{C}, \quad \mu > 0.$$

Further,

$$J^{\mu}_{n-2\lambda,\lambda}(z) = v_0(n;\lambda,\mu)(z/2)^n \left(1 + \sum_{k=1}^{\infty} (-1)^k\, \frac{v_k(n;\lambda,\mu)}{v_0(n;\lambda,\mu)}(z/2)^{2k}\right)$$

$$= v_0(n;\lambda,\mu)(z/2)^n \left(1 + \frac{v_1(n;\lambda,\mu)}{v_0(n;\lambda,\mu)} \sum_{k=1}^{\infty} (-1)^k\, \frac{v_k(n;\lambda,\mu)}{v_1(n;\lambda,\mu)}(z/2)^{2k}\right).$$

We transform the last sum, after introducing a denotation

$$\frac{v_k(n;\lambda,\mu)}{v_1(n;\lambda,\mu)} = \frac{\Gamma(\lambda+2)}{\Gamma(\lambda+k+1)} \cdot \frac{\Gamma(n+\mu-\lambda+1)}{\Gamma(n+k\mu-\lambda+1)}.$$

Using again the Stirling formula, we have

$$\lim_{n\to\infty} \left(\frac{\Gamma(n+\mu-\lambda+1)}{\Gamma(n+k\mu-\lambda+1)} n^{\mu}\right) = \begin{cases} 1 & \text{for } k=2 \\ 0 & \text{for } k=3,4,5,\ldots \end{cases}$$

whence we conclude that $\lim\limits_{n\to\infty} \left(\frac{\Gamma(n+\mu-\lambda+1)}{\Gamma(n+k\mu-\lambda+1)}\right) = 0$ for $k = 2, 3, 4, \ldots$

Denoting

$$\theta^{\mu}_{n-2\lambda,\lambda}(z) = \sum_{k=1}^{\infty} (-1)^k\, \frac{v_k(n;\lambda,\mu)}{v_0(n;\lambda,\mu)}(z/2)^{2k}$$

and having in mind that

$$\lim_{n\to\infty}\left(\frac{v_1(n;\lambda,\mu)}{v_0(n;\lambda,\mu)}\right)=0,$$

the rest immediately follows the same way as in Theorem 2.3.1. $\square$

Remark 2.3.2. *Let* $\mu>0$. *Then the asymptotic formula for the generalized Lommel–Wright functions follows in the same way and it is the following*

$$J^{\mu,m}_{n-2\lambda,\lambda}(z)=\frac{(z/2)^n}{\Gamma^m(\lambda+1)\Gamma(n-\lambda+1)}(1+\theta^{\mu,m}_{n-2\lambda,\lambda}(z)),\quad z\in\mathbb{C}\,,\tag{2.3.5}$$

$$\theta^{\mu,m}_{n-2\lambda,\lambda}(z)\to 0\ \ as\ \ n\to\infty\quad(n\in\mathbb{N}),$$

holds. The functions $\theta^{\mu,m}_{n-2\lambda,\lambda}(z)$ *are holomorphic functions of* z *in* $\mathbb{C}$. *The convergence is uniform on the compact subsets of the complex plane* $\mathbb{C}$.

Remark 2.3.3. *Of course, the asymptotic formulae* (2.3.4) *and* (2.3.5) *for the functions* $J^{\mu}_{n-2\lambda,\lambda}(z)$, *given by* (2.3.3), *and* $J^{\mu,m}_{n-2\lambda,\lambda}(z)$ *are found under condition that the first coefficients in* (2.3.3) *are different from zero, elsewhere it should be a little refined. In what follows in this book we are under this condition.*

2.4 Asymptotics for Lommel and Struve functions

As previously mentioned (see end of Section 6.1), the generalized Bessel–Maitland functions (1.4.2) turn into Lommel functions $s_{\alpha,\nu}$ [7,

2, p.50, (69)], for parameter $\mu = 1$. This correlation (see [26], p.110, (8.3); [12], p. 352-353) can be rewritten as

$$s_{\alpha,\nu}(z) = 2^{\alpha-1}\Gamma(\frac{\alpha-\nu+1}{2})\Gamma(\frac{\alpha+\nu+1}{2})\, J^1_{\nu,\frac{\alpha-\nu+1}{2}}(z), \tag{2.4.1}$$

or by taking $\lambda = \dfrac{\alpha-\nu+1}{2}$, in the form

$$s_{\alpha,\alpha+1-2\lambda}(z) = 2^{\alpha-1}\Gamma(\lambda)\Gamma(\alpha+1-\lambda)\, J^1_{\alpha+1-2\lambda,\lambda}(z). \tag{2.4.2}$$

Additionally, if $\lambda = 1/2$ that is $\alpha = \nu$, we obtain from (2.4.2) the Struve functions [7, **2**, p.51, (84)]:

$$\mathbf{H}_\nu(z) = \frac{2^{1-\nu}}{\sqrt{\pi}\Gamma(\nu+1/2)}\, s_{\nu,\nu}(z) = J^1_{\nu,1/2}(z). \tag{2.4.3}$$

Then, Theorem 2.3.2 provides the following corollaries.

Corollary 2.4.1. *For Lommel functions of form* (2.4.2) *the asymptotic formula*

$$s_{m,m+1-2\lambda}(z) = 2^{m-1}\Gamma(\lambda)\Gamma(m+1-\lambda)\, J^1_{m+1-2\lambda,\lambda}(z)$$

$$= [4\lambda(m+1-\lambda)]^{-1}\, z^{m+1}\, \left(1+\theta^1_{m+1-2\lambda,\lambda}(z)\right), \tag{2.4.4}$$

holds, with

$$\theta^1_{m+1-2\lambda,\lambda}(z) \to 0 \ \ as \ \ m \to \infty \quad (m \in \mathbb{N}),$$

the convergence being uniform on the compact subsets of complex z-plane.

Corollary 2.4.2. *For the Struve functions* (2.4.3)*, the following asymptotic formula follows from* (2.3.4) *:*

$$H_n(z) = J^1_{n,1/2}(z) = \frac{2(z/2)^n}{\sqrt{\pi}\Gamma(n+1/2)}\left(1+\theta^1_{n,1/2}(z)\right), \tag{2.4.5}$$

holds, with

$$\theta^1_{n,1/2}(z) \to 0 \;\; as \;\; n \to \infty \quad (n \in \mathbb{N}),$$

the convergence being uniform on the compact subsets of complex z-plane.

2.5 3-D representations

In this section we give examples of 3-dimensional graphs of generalized Bessel–Maitland functions.

In the first two figures, some functions $J^\mu_\nu(z)$ are shown for values of $z = x$ on the interval [-90, 90]. The cases $\mu = 1$, $\mu = 2$, $\mu = 3$ are illustrated. Namely, in Figure 1 and Figure 2 we show the graphics of the functions with indices $0 \leq \nu \leq 13$ and $28 \leq \nu \leq 36$, respectively. In Figure 3 some functions of the type $J^\mu_{\nu-2\lambda,\lambda}(z)$ are illustrated with indices $\nu \in [0, 17]$ and for $z = x$ on the interval [0, 90]. In view of relation (2.4.2), the first two pictures ($\mu = 1$) give 3-D representations of Lommel functions of the forms $s_{\nu+1,\nu}(x)$ and $s_{\nu+3,\nu}(x)$, respectively.

Note that, in general speaking, this chapter is mainly based on the results, obtained in [37].

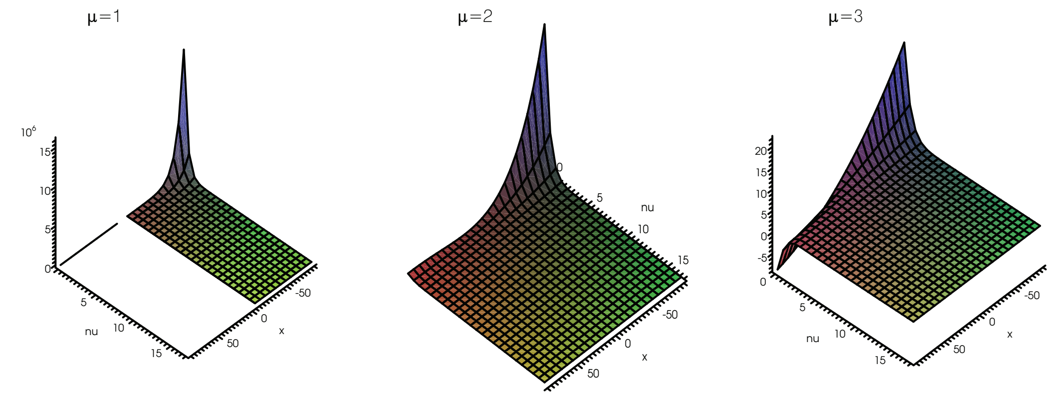

Figure 2.5.1

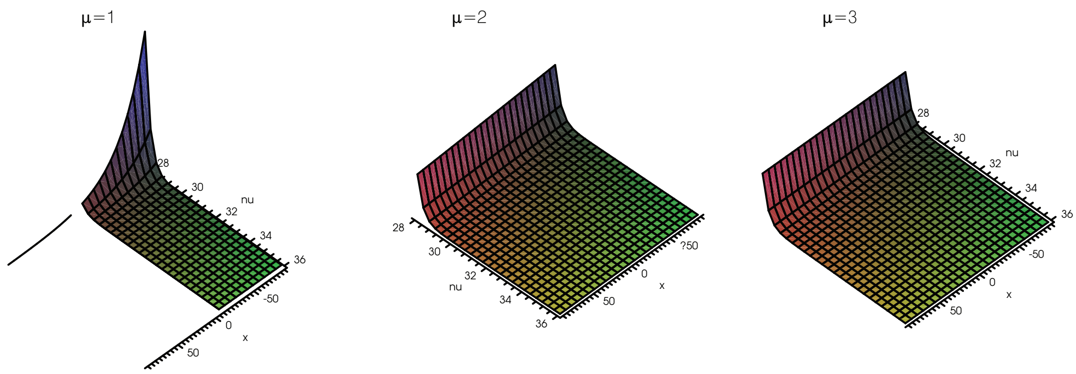

Figure 2.5.2

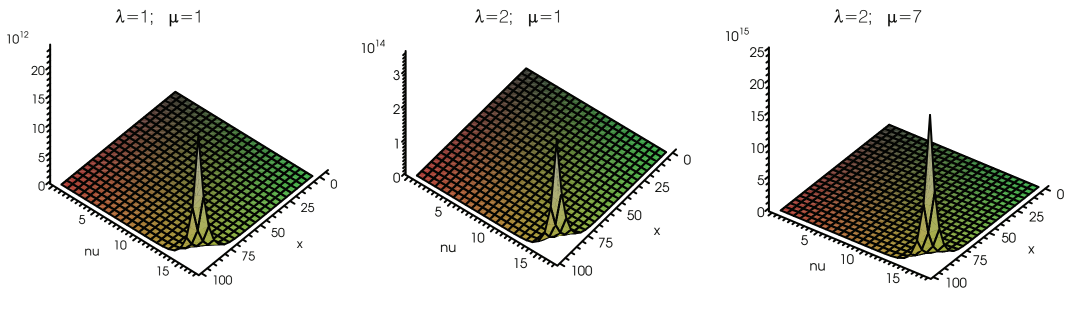

Figure 2.5.3

3 Bessel Series

3.1 Introduction

As well known, the studies on the properties of the complex functions, which are holomorphic in a given region of the complex plane, are often based on the possibility of their representations by series in some particular countable systems of functions, holomorphic in the considered region. For circle domains, most often the Taylor systems are used, leading to expansions by power series. The classical orthogonal polynomials are also often used. Series in the system of Bessel functions of first kind with nonnegative indices have been considered since 19th century by Neumann. To him is due the representation of Cauchy's kernel by bilinear series in these functions and in Neumann's 'polynomials'. Some important properties of the power series in a complex domain are given by the classical Cauchy–Hadamard, Abel and Tauber theorems. Theorems of this type are proven for several systems of orthogonal polynomials and entire functions [53], [54], see also [52]. For this purpose some formulae are used which describe the behaviour of the functions for ‘large’ values of the indices. Therefore, it is useful to know their representations in the case when indices grow to infinity.

3.2 Classical results for the power series

An important property of the holomorphic functions is their possibility to be expand by a power series

$$\sum_{n=0}^{\infty} a_n z^n, \quad a_n \in \mathbb{C}, \quad n = 0, 1, 2, ... \tag{3.2.1}$$

Some useful information on the convergence of such a type of series in complex domains is given by the classical Cauchy–Hadamard, Abel and Tauber theorems. Such a type of results were also obtained by the famous Bulgarian academician N. Obrechkoff [29]–[31].

So, by the Cauchy–Hadamard theorem, each series of the kind (3.2.1) is absolutely convergent in the disk $D(0; R)$ with a radius $R = \left(\limsup_{n\to\infty} |a_n|^{1/n}\right)^{-1}$ and divergent on its outside $|z| > R$. In general, by Abel's theorem, from the convergence of a power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ at a point z_0 it follows the existence of the limit $\lim_{z\to z_0} f(z) = f(z_0)$, when z belongs to a suitable angle domain with a vertex at a point z_0. The geometrical series [64, p. 92]:

$$\frac{1}{1+z} = 1 - z + z^2 - z^3 + \ldots,$$

at the point $z_0 = 1$ gives an example that in general, the inverse proposition is not true, i.e. the existence of the limit, mentioned above, does not imply the convergence of the series $\sum\limits_{n=0}^{\infty} a_n z_0^n$ without additional conditions on the growth of the coefficients.

The corresponding result in this direction is given by the following theorem [8, Th. 85].

Theorem 3.2.1 (of Tauber). *If the coefficients of the power series satisfy the condition*

$$\lim_{n\to\infty} n a_n = 0$$

and if

$$\lim_{z\to 1} f(z) = S \quad (z \to 1 \textit{ radially}),$$

then the series $\sum a_n$ *is convergent and*

$$\sum_{n=0}^{\infty} a_n = S.$$

As a matter of fact, the Tauber theorem has already more than 100 years history. Recently J. Korevaar has published his survey-paper [22] devoted to the Century Anniversary of complex Tauberian theory.

3.3 Bessel series

Let $J_\nu(z)$ denote the Bessel functions, defined with (1.1.2). Let us consider the series, defined by means of the Bessel functions, of the type

$$\sum_{n=0}^{\infty} a_n J_n(z), \quad z \in \mathbb{C} \tag{3.3.1}$$

and briefly call them *Bessel series*.

In this chapter we study their geometry of convergence, more precisely, we determine where these series converge and where they do not, and moreover, where the convergence is uniform and where it is not. Their disks of convergence have been found and studied the behaviour on the boundaries of these domains, proving theorems of Cauchy–Hadamard, and Abel type. The definitions and main statements, concerning the above mentioned results, have first appeared in Paneva-Konovska [34, 36]. The asymptotic formulae, obtained for the Bessel functions in Chapter 2, in the cases of 'large' values of indices, are used for proving the convergence theorems for the considered series.

3.4 Cauchy–Hadamard type theorem

In this section the domain of convergence of the Bessel series (3.3.1) is found and proved the corresponding Cauchy–Hadamard type theorem.

Theorem 3.4.1 (of Cauchy–Hadamard type [34]). *The domain of convergence of the series* (3.3.1) *is the disk* $D(0; R)$ *with a radius of* *convergence*

$$R = 2\left(\limsup_{n\to\infty}(\ |a_n|\ /\ \Gamma(n+1)\ )^{1/n}\right)^{-1}. \qquad (3.4.1)$$

More precisely, the series (3.3.1) *is absolutely convergent in the disk* $D(0;R)$ *and divergent in the domain* $|z| > R$. *The cases* $R = 0$ *and* $R = \infty$ *are incorporated in the common case.*

Proof. For convenience, let us denote

$$u_n(z) = a_n J_n(z),\ b_n = 2^{-1}(\ |a_n|\ /\ \Gamma(n{+}1)\ )^{1/n},\ \Lambda = 1/R = \limsup_{n\to\infty} b_n.$$

Using the asymptotic formula (2.1.3), we get

$$u_n(z) = a_n(z/2)^n(1+\theta_n(z))\ /\ \Gamma(n+1).$$

The proof goes in three cases.

1. $\Lambda = 0$, then $\lim\limits_{n\to\infty} b_n = \limsup\limits_{n\to\infty} b_n = 0$. Let us fix $z \neq 0$. Obviously, there exists a number N_1 such that for every $n > N_1$ the inequalities $|1+\theta_n(z)| < 2$ and $2b_n < 1/|z|$ hold, whence $|u_n(z)| = b_n^n|z|^n|1+\theta_n(z)| < 2^{1-n}$. The absolute convergence of (3.3.1) follows immediately from this inequality.

2. $0 < \Lambda < \infty$. First, let z be in the domain $D(0;R)$, i.e. $|z|/R < 1$. Then $\limsup\limits_{n\to\infty} |z|b_n < 1$. Therefore, a number $q < 1$ exists such that $\limsup\limits_{n\to\infty} |z|b_n \le q$, whence $|z|^n b_n^n \le q^n$. Using the asymptotic formula for the general term $u_n(z)$ of the series (3.3.1), we obtain $|u_n(z)| = b_n^n|z|^n|1+\theta_n(z)| \le q^n|1+\theta_n(z)|$. Since $\lim\limits_{n\to\infty}\theta_n(z) = 0$, there

exists N_2 such that $|1 + \theta_n(z)| < 2$ for every $n > N_2$, and hence $|u_n(z)| \leq 2q^n$. As the series $\sum\limits_{n=0}^{\infty} 2q^n$ is convergent, the series (3.3.1) is also convergent, even absolutely.

Now, let z lie outside this domain, i.e. $|z|/R > 1$. Then $\limsup\limits_{n\to\infty} |z|b_n > 1$ and therefore there exist infinite number of values n_k of n with the property $|z|^{n_k} b_{n_k}^{n_k} > 1$. Since $\lim\limits_{n\to\infty} \theta_n(z) = 0$, there exists N_3 so that for $n_k > N_3$; $|1 + \theta_{n_k}(z)| \geq 1/2$, i.e. $|u_{n_k}(z)| \geq 1/2$ for infinite number of values of n. This means that the necessary condition for convergence is not satisfied and therefore the series (3.3.1) is divergent.

3. $\Lambda = \infty$. Let $z \in \mathbb{C}\backslash\{0\}$. Then $b_{n_k} > 1/|z|$ for infinite number of values n_k of n, whence $|u_{n_k}(z)| = |z|^{n_k}\ b_{n_k}^{n_k}\ |1 + \theta_{n_k}(z)| \geq 1/2$. In other terms, the necessary condition for the convergence of the series (3.3.1) is not satisfied, and we deduce that the series (3.3.1) is divergent for every $z \neq 0$. □

Corollary 3.4.1. *Let the series* (3.3.1) *converges at the point* $z_0 \neq 0$. *Then it is absolutely convergent in the disk* $D(0; |z_0|)$. *Inside the disk* $D(0; R)$*, i.e. on each closed disk* $|z| \leq r$ *(*$r < R$*), the convergence is uniform.*

Proof. Indeed, since the considered series converges at the point $z_0 \neq 0$, then its radius of convergence R is a positive number, and moreover the point z_0 lies either in the disk $D(0; R)$ or on its boundary - the circle $C(0; R)$. That is why, the disk $D(0; |z_0|)$ is either a part of the domain of convergence or it coincides with it, whence the absolute convergence follows. To prove uniformity of the convergence inside the disk $D(0;R)$, it is sufficient to show that the series is uniformly convergent on each closed disk $|z| \leq r$ $(r < R)$. To this purpose, choosing a point ζ, $|\zeta| = \rho$, $r < \rho < R$ and considering the series (3.3.1), we estimate $|a_n J_n(z)|$. First, mention that some of the values of $J_n(\zeta)$, but only finite numbers of them, can be zero. Then, having in view (2.1.3), as well, there exists a number p, such that the expression $|a_n J_n(z)|$ can be written as follows

$$|a_n J_n(z)| = |a_n J_n(\zeta)| \frac{|J_n(z)|}{|J_n(\zeta)|}$$

$$= |a_n J_n(\zeta)| \, \frac{|z^n||1+\theta_n(z)|}{|\zeta^n||1+\theta_n(\zeta)|} \leq |a_n J_n(\zeta)| \, \frac{|1+\theta_n(z)|}{|1+\theta_n(\zeta)|}$$

for all $n > p$ and $|z| \leq r$.

Because of (2.2.1) and the relation $\lim\limits_{n\to\infty} \dfrac{1}{n+1} = 0$, we obtain the equalities $\lim\limits_{n\to\infty}(1+\theta_n(z)) = 1$, $\lim\limits_{n\to\infty}(1+\theta_n(\zeta))^{-1} = 1$. Therefore, there exists a number A such that $|1+\theta_n(z)||1+\theta_n(\zeta)|^{-1} \leq A$ and hence $|a_n J_n(z)| \leq A|a_n J_n(\zeta)|$, for all the values of $n > p$ and $|z| \leq r$. Since the series $\sum\limits_{n=0}^{\infty} a_n J_n(\zeta)$ is absolutely convergent and by the Weierstrass criterium for the uniform convergence, the proof is completed. $\square$

The very disk of convergence is not obligatorily a domain of uniform convergence and on its boundary the series may even be divergent.

3.5 Abel type theorem

It turns out that the Abel theorem fails even for series of the kind $\sum\limits_{k=1}^{\infty} a_{n_k} z^{n_k}$, where $(n_1, n_2, \ldots, n_k, \ldots)$ is a suitable permutation of the nonnegative integers [64, p.92]. Therefore, it is interesting to know if for series in a given sequence of holomorphic functions a statement like the Abel theorem is available. A positive answer to this question, concerning the series in Laguerre and Hermite polynomials, is given by Rusev in his monographs [53, Ch. 11, §11.3] and [54, Ch. 4, §4].

Let $z_0 \in \mathbb{C}$, $0 < R < \infty$, $|z_0| = R$ and g_φ be an arbitrary angular domain with a size $2\varphi < \pi$ and a vertex at the point $z = z_0$, that is symmetric with respect to the line passing through the points 0 and z_0, and d_φ be the part of the angular domain g_φ, closed between the angle's arms and the arc of the circle centred at the origin and touching the arms of the angle. The following theorem refers to the limit of the sum of (3.3.1) at the point z_0, provided $z \in g_\varphi$.

Theorem 3.5.1 (of Abel type). *Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of complex numbers, R be the positive number defined by* (3.4.1), *$F(z)$ be the sum of the series* (3.3.1) *on the domain $D(0;R)$, i.e.*

$$F(z) = \sum_{n=0}^{\infty} a_n J_n(z), \quad z \in D(0;R), \tag{3.5.1}$$

and this series converges at the point z_0 of the boundary of $D(0;R)$. Then:

(I) *The series* (3.3.1) *is uniformly convergent on the domain* d_φ.

(II) *The following relation holds*

$$\lim_{z \to z_0} F(z) = \sum_{n=0}^{\infty} a_n J_n(z_0), \tag{3.5.2}$$

provided $z \in D(0; R) \cap g_\varphi$.

Proof. (I) To prove the uniform convergence we use the following geometrical inequality

$$|z - z_0| \cos \varphi < 2(|z_0| - |z|), \tag{3.5.3}$$

that is the crucial point of the proof.

So, let $z \in d_\varphi$. Setting

$$\begin{gathered} S_k(z) = \sum_{n=0}^{k} a_n J_n(z), \\ S_k(z_0) = \sum_{n=0}^{k} a_n J_n(z_0), \quad \lim_{k \to \infty} S_k(z_0) = s, \\ \beta_n = S_n(z_0) - s, \quad \beta_n - \beta_{n-1} = a_n J_n(z_0), \end{gathered} \tag{3.5.4}$$

we obtain

$$S_{k+p}(z) - S_k(z) = \sum_{n=0}^{k+p} a_n J_n(z) - \sum_{n=0}^{k} a_n J_n(z) = \sum_{n=k+1}^{k+p} a_n J_n(z).$$

According to Remark 2.2.2, there exists a natural number N_0 such that $J_n(z_0) \neq 0$ when $n > N_0$. Let $k > N_0$ and $p > 0$. Then, using the denotation

$$\gamma_n(z;z_0) = J_n(z)/J_n(z_0),$$

we can write the difference $S_{k+p}(z) - S_k(z)$ as follows:

$$S_{k+p}(z) - S_k(z) = \sum_{n=k+1}^{k+p} a_n J_n(z_0) \frac{J_n(z)}{J_n(z_0)} = \sum_{n=k+1}^{k+p} a_n J_n(z_0) \gamma_n(z;z_0).$$

Now, by the Abel transformation (see in [27, Vol.1, Ch.1, p.32, 3.4:7]), we obtain subsequently:

$$S_{k+p}(z) - S_k(z) = \sum_{n=k+1}^{k+p} (\beta_n - \beta_{n-1})\gamma_n(z;z_0)$$

$$= \beta_{k+p}\gamma_{k+p}(z;z_0) - \beta_k\gamma_{k+1}(z;z_0) - \sum_{n=k+1}^{k+p-1} \beta_n(\gamma_{n+1}(z;z_0) - \gamma_n(z;z_0)),$$

$$S_{k+p}(z) - S_k(z) = (S_{k+p}(z_0) - s)\gamma_{k+p}(z;z_0) - (S_k(z_0) - s)\gamma_{k+1}(z;z_0)$$
$$+ \sum_{n=k+1}^{k+p-1} (S_n(z_0) - s) \times \left(\frac{J_n(z)}{J_n(z_0)} - \frac{J_{n+1}(z)}{J_{n+1}(z_0)} \right).$$

So, using the last relation, we are going to estimate the module of the difference $S_{k+p}(z) - S_k(z)$ as follows:

$$|S_{k+p}(z) - S_k(z)| \le |S_{k+p}(z_0) - s||\gamma_{k+p}(z;z_0)| + |S_k(z_0) - s||\gamma_{k+1}(z;z_0)|$$
$$+ \sum_{n=k+1}^{k+p-1} |S_n(z_0) - s| \times \left| \frac{J_n(z)}{J_n(z_0)} - \frac{J_{n+1}(z)}{J_{n+1}(z_0)} \right|. \qquad (3.5.5)$$

Because of (2.2.1) and the relations $\lim\limits_{n\to\infty} \dfrac{1}{n+1} = 0$, $\lim\limits_{n\to\infty} (1+\theta_n(z_0))^{-1} = 1$, there exist numbers A and $N_1 > N_0$ such that $|1+\theta_n(z)| \le A/2$ for all the natural values of n and $|1+\theta_n(\zeta)|^{-1} < 2$ for $n > N_1$, whence

$$|\gamma_n(z, z_0)| \le A \quad \text{for } n > N_1. \tag{3.5.6}$$

Further, setting

$$j_n(z, z_0) = \frac{J_n(z)}{J_n(z_0)} - \frac{J_{n+1}(z)}{J_{n+1}(z_0)} = \frac{z^n}{z_0^n} \times \left(\frac{1+\theta_n(z)}{1+\theta_n(z_0)} - \frac{z}{z_0} \times \frac{1+\theta_{n+1}(z)}{1+\theta_{n+1}(z_0)} \right)$$

and observing that $j_n(z_0, z_0) = 0$, we apply the Schwarz lemma, named after Hermann Amandus Schwarz, for $j_n(z, z_0)$. Thus, we get that there exists a constant C such that:

$$|j_n(z, z_0)| = |J_n(z)/J_n(z_0) - J_{n+1}(z)/J_{n+1}(z_0)| \le C|z - z_0||z/z_0|^n,$$

whence, and in accordance with (3.5.3):

$$\sum_{n=k+1}^{k+p+1} |j_n(z, z_0)| \le \sum_{n=0}^{\infty} C|z - z_0||z/z_0|^n = C|z_0| \times \frac{|z - z_0|}{|z_0| - |z|} < \frac{2C|z_0|}{\cos\varphi}. \tag{3.5.7}$$

Let ε be an arbitrary positive number. Taking in view the third of the relations (3.5.4), we can confirm that there exists a positive number $N_2 > N_0$ so large that

$$|S_n(z_0) - s| < \min\left(\frac{\varepsilon}{3A}, \frac{\varepsilon \cos\varphi}{6C|z_0|}\right) \quad \text{for } n > N_2. \tag{3.5.8}$$

Now, let $N = N(\varepsilon) = \max(N_1, N_2)$ and $k > N$. Therefore (3.5.5)–(3.5.8) give

$$|S_{k+p}(z) - S_k(z)| < \frac{2\varepsilon}{3} + \frac{\varepsilon\cos\varphi}{6C|z_0|} \times \sum_{n=k+1}^{k+p+1} |j_n(z, z_0)|$$

$$< \frac{2\varepsilon}{3} + \frac{\varepsilon\cos\varphi}{6C|z_0|} \times \frac{2C|z_0|}{\cos\varphi} = \varepsilon,$$

that completes the proof of (I).

(II) The second part of the theorem could be proved in a similar way, estimating the module of the second summand of the difference

$$\Delta(z) = \sum_{n=0}^{\infty} a_n J_n(z_0) - F(z)$$

$$= \sum_{n=0}^{k} a_n(J_n(z_0) - J_n(z)) + \sum_{n=k+1}^{\infty} a_n(J_n(z_0) - J_n(z)),$$

'near' the vertex of the angle domain g_φ in the part d_φ. However, here we give another proof as a corollary of the first part. Namely, the uniform convergence of the series along with the equalities $\lim\limits_{z\to z_0} J_n(z) = J_n(z_0)$ ($n \in \mathbb{N}_0$) verify the equality (3.5.2) that completes the proof for the considered series. □

Remark 3.5.1. *If the series* (3.3.1) *has a finite and non-zero radius of convergence R, it converges at the point $z_0 \in C(0; R)$ and F is the* *holomorphic function defined by this series in its domain of convergence $D(0;R)$, then by the Theorem 3.5.1 it follows that*

$$\lim_{z\to z_0,\ z\in d_\varphi} F(z) = F(z_0),$$

i.e. the restriction of the function F to each set of the kind d_φ is continuous at the point z_0.

3.6 Tauber type theorem

Let us consider the series $\sum\limits_{n=0}^{\infty} a_n,\ a_n \in \mathbb{C}$. Let $z_0 \in \mathbb{C}\setminus\mathbb{R}$ and $|z_0| = R$. Since all the zeros of $J_n(z)$ are real, then $J_n(z_0) \neq 0$. Now, for the sake of brevity, we denote

$$J_n^*(z;z_0) = \frac{J_n(z)}{J_n(z_0)}, \quad z \in \mathbb{C}. \tag{3.6.1}$$

Let the series

$$F(z) = \sum_{n=0}^{\infty} a_n J_n^*(z;z_0) \tag{3.6.2}$$

be convergent for $|z| < R$. Then the next theorem is valid.

Theorem 3.6.1 (of Tauber type)**.** *If there exists*

$$\lim_{z\to z_0} F(z) = S \quad (|z| < R, z \to z_0 \textit{ radially})$$

and

$$\lim_{n\to\infty} na_n = 0, \tag{3.6.3}$$

then the series $\sum\limits_{n=0}^{\infty} a_n$ *is convergent and*

$$\sum_{n=0}^{\infty} a_n = S.$$

Proof. Taking a point z of the segment $[0, z_0]$, we have

$$\sum_{n=0}^{k} a_n - F(z) = \sum_{n=0}^{k} a_n - \sum_{n=0}^{\infty} a_n J_n^*(z; z_0)$$

$$= \sum_{n=0}^{k} a_n \frac{J_n(z_0)}{J_n(z_0)} - \sum_{n=0}^{\infty} a_n \frac{J_n(z)}{J_n(z_0)} = \sum_{n=0}^{k} a_n \frac{J_n(z_0) - J_n(z)}{J_n(z_0)} - \sum_{n=k+1}^{\infty} a_n \frac{J_n(z)}{J_n(z_0)}$$

and therefore,

$$|\sum_{n=0}^{k} a_n - F(z)| \leq \sum_{n=0}^{k} |a_n| \left| \frac{J_n(z_0) - J_n(z)}{J_n(z_0)} \right| + \sum_{n=k+1}^{\infty} |a_n| \left| \frac{J_n(z)}{J_n(z_0)} \right|. \tag{3.6.4}$$

By using the asymptotic formula (2.1.3) for the Bessel functions of first kind, we obtain:

$$a_n \frac{J_n(z)}{J_n(z_0)} = a_n \left(\frac{z}{z_0}\right)^n \frac{1 + \theta + n(z)}{1 + \theta_n(z_0)} = a_n \left(\frac{z}{z_0}\right)^n \left(1 + \widetilde{\theta}_n(z; z_0)\right).$$

Let ε be an arbitrary positive number. We choose a number N_1 so large that the inequalities $|1 + \widetilde{\theta}_k(z; z_0)| < 2, |ka_k| < \frac{\varepsilon}{6}$ hold as $k \geq N_1$. If $k > N_1$ and z is on the segment $[0, z_0]$, then for the second summand in (3.6.4) the following estimate is valid:

$$\sum_{n=k+1}^{\infty} |a_n| \left| \frac{J_n(z)}{J_n(z_0)} \right| = \sum_{n=k+1}^{\infty} |a_n| \left| \frac{z}{z_0} \right|^n |1 + \widetilde{\theta}_n(z; z_0)|$$

$$\le 2\left|\frac{z}{z_0}\right|^{k+1}\sum_{n=k+1}^{\infty}|a_n|\left|\frac{z}{z_0}\right|^{n-k-1}\le\sum_{n=0}^{\infty}|a_{n+k+1}|\left|\frac{z}{z_0}\right|^{n} \tag{3.6.5}$$

$$= 2\sum_{n=0}^{\infty}\frac{|(n+k+1)a_{n+k+1}|}{n+k+1}\left|\frac{z}{z_0}\right|^{n} < 2\sum_{n=0}^{\infty}\frac{\varepsilon/6}{n+k+1}\left|\frac{z}{z_0}\right|^{n}$$

$$< \frac{2}{k}\frac{\varepsilon}{6}\frac{1}{1-|z/z_0|}=\frac{\varepsilon}{3}\frac{1}{k}\frac{|z_0|}{|z_0|-|z|}\ .$$

Now let us consider the first summand in (3.6.4). We have:

$$\sum_{n=0}^{k}|a_n|\left|\frac{J_n(z_0)-J_n(z)}{J_n(z_0)}\right|=\sum_{n=0}^{m}|a_n|\left|\frac{J_n(z_0)-J_n(z)}{J_n(z_0)}\right|+\sum_{n=m+1}^{k}|a_n|$$

According to Schwarz's lemma, there exists a constant C such that

$$\left|\frac{J_n(z_0)-J_n(z)}{J_n(z_0)}\right|< C|z-z_0|.$$

Moreover, there exists a number N_2 such that the following inequality

$$\sum_{n=0}^{m}|a_n|\left|\frac{J_n(z_0)-J_n(z)}{J_n(z_0)}\right|\le C\,|z-z_0|\,k\,\frac{\sum_{n=0}^{m}|a_n|}{k}< C\,|z-z_0|\,k\,\frac{\varepsilon}{3RC}=|z-z_0|\,k\,\frac{\varepsilon}{3R}. \tag{3.6.6}$$

holds as $k>N_2$. It remains to estimate the sum

$$\sum_{n=m+1}^{k}|a_n|\left|\frac{J_n(z_0)-J_n(z)}{J_n(z_0)}\right|.$$

To this end, using asymptotic formula (2.1.3) for the Bessel functions of first kind, we find consequently:

$$\frac{J_n(z_0)-J_n(z)}{J_n(z_0)}=\frac{(z_0)^n(1+\theta_n(z_0))-z^n(1+\theta_n(z))}{z_0^n(1+\theta_n(z_0))}$$

$$= 1 - \left(\frac{z}{z_0}\right)^n \frac{1+\theta_n(z)}{1+\theta_n(z_0)} = 1 - \left(\frac{z}{z_0}\right)^n \left[1 + \frac{\theta_n(z)-\theta_n(z_0)}{1+\theta_n(z_0)}\right]$$

$$= 1 - \left(\frac{z}{z_0}\right)^n - \left(\frac{z}{z_0}\right)^n \frac{\theta_n(z)-\theta_n(z_0)}{1+\theta_n(z_0)} .$$

Therefore,

$$\left|\frac{J_n(z_0)-J_n(z)}{J_n(z_0)}\right| \le \left|1 - \left(\frac{z}{z_0}\right)^n\right| + \left|\frac{z}{z_0}\right|^n \left|\frac{\theta_n(z)-\theta_n(z_0)}{1+\theta_n(z_0)}\right| . \qquad (3.6.7)$$

We obtain the following inequalities

$$\left|1 - \left(\frac{z}{z_0}\right)^n\right| = \left|1 - \frac{z}{z_0}\right| \left|1 + \frac{z}{z_0} + \left(\frac{z}{z_0}\right)^2 + \cdots + \left(\frac{z}{z_0}\right)^{n-1}\right| \le n \left|1 - \frac{z}{z_0}\right|$$

for the first summand of (3.6.7). According to Schwarz's lemma, there exists a constant ρ such that

$$\left|\frac{\theta_n(z)-\theta_n(z_0)}{1+\theta_n(z_0)}\right| \le 1 \qquad \text{as} \qquad |z-z_0| < \rho.$$

Then, for such $|z|$, we obtain for the second summand of (3.6.7):

$$\left|\frac{z}{z_0}\right|^n \left|\frac{\theta_n(z)-\theta_n(z_0)}{1+\theta_n(z_0)}\right| \le \left|\frac{z}{z_0}\right|^n |z-z_0|.$$

From (3.6.3) it follows that

$$\lim_{n\to\infty} a_n = 0, \quad \lim_{k\to\infty} \frac{\sum\limits_{n=1}^{k} n|a_n|}{k} = 0, \quad \lim_{k\to\infty} \frac{\sum\limits_{n=1}^{k} |a_n|}{k} = 0.$$

Then a number N_3 exists such that

$$\frac{\sum\limits_{n=m+1}^{k} n|a_n|}{k} < \frac{\varepsilon}{3(1+R)} \qquad \text{and} \qquad \frac{\sum\limits_{n=m+1}^{k} |a_n|}{k} < \frac{\varepsilon}{3(1+R)} \qquad \text{as} \qquad k > N_3.$$

Therefore,

$$\sum_{n=m+1}^{k} |a_n| \left|\frac{J_n(z_0) - J_n(z)}{J_n(z_0)}\right| \leq \sum_{n=m+1}^{k} n|a_n| \left|1 - \frac{z}{z_0}\right| + \sum_{n=m+1}^{k} |a_n| \left|\frac{z}{z_0}\right|^n |z - z_0|$$

$$\leq k\,\frac{|z-z_0|}{R}\,\frac{\sum\limits_{n=m+1}^{k} n|a_n|}{k} + k\,|z-z_0|\,\frac{\sum\limits_{n=m+1}^{k} |a_n|}{k} \tag{3.6.8}$$

$$< k\,|z-z_0|\,\frac{1+R}{R}\,\frac{\varepsilon}{3}(1+R) = k\,|z-z_0|\,\frac{\varepsilon}{3R}.$$

Finally, let us note that

$$\left|\sum_{n=0}^{k} a_n - F(z)\right| \leq \sum_{n=0}^{m} |a_n| \left|\frac{J_n(z_0) - J_n(z)}{J_n(z_0)}\right|$$

$$+ \sum_{n=m+1}^{k} |a_n| \left|\frac{J_n(z_0) - J_n(z)}{J_n(z_0)}\right| + \sum_{n=k+1}^{\infty} |a_n| \left|\frac{J_n(z)}{J_n(z_0)}\right| .$$

Let $N = \max(N_1, N_2, N_3), k > N$ and $|z - z_0| < \rho$. Then by using (3.6.5), (3.6.6), (3.6.8), we can conclude that

$$\left|\sum_{n=0}^{k} a_n - F(z)\right| < |z-z_0|\,k\,\frac{\varepsilon}{3R} + k\,|z-z_0|\,\frac{\varepsilon}{3R} + \frac{\varepsilon}{3}\,\frac{1}{k}\,\frac{|z_0|}{|z_0| - |z|}$$

$$= \frac{\varepsilon}{3} \left[\frac{2k}{R} |z - z_0| + \frac{1}{k} \frac{|z_0|}{|z_0| - |z|} \right] .$$

If we substitute z by $z_0(1 - \frac{1}{k})$, then

$$\left| \sum_{n=0}^{k} a_n - F\left(z_0(1 - |frac1k)\right) \right| < \frac{\varepsilon}{3} 3 = \varepsilon.$$

This proves that $\lim\limits_{k\to\infty} \sum\limits_{n=0}^{k} a_n$ exists and equals $\lim\limits_{k\to\infty} F\left(z_0(1 - \frac{1}{k})\right)$, i.e.

$$\sum_{n=0}^{\infty} a_n = \lim_{k\to\infty} F\left(z_0(1 - \frac{1}{k})\right) = S.$$

Thus the theorem is proved. $\square$

4 Bessel Type Series

4.1 Introduction

Consider again the Bessel type functions (1.4.1), (1.4.2) and (1.4.3) with more additional indices. In this Chapter we consider series in such functions and call them *Bessel type series*. We prove the corresponding Cauchy–Hadamard, Abel and Tauber type theorems for them.

Let us begin with the series of the kind

$$\sum_{n=0}^{\infty} a_n \widetilde{J}_n^{\mu}(z), \quad \widetilde{J}_n^{\mu}(z) = z^n J_n^{\mu}(z), \quad z \in \mathbb{C}, \quad \mu > 0, \tag{4.1.1}$$

with complex coefficients a_n and continue with the series of the kind

$$\sum_{n=0}^{\infty} a_n J_{n-2\lambda,\lambda}^{\mu}(z), \quad z \in \mathbb{C}, \quad \mu > 0, \tag{4.1.2}$$

respectively

$$\sum_{n=0}^{\infty} a_n J_{n-2\lambda,\lambda}^{\mu,m}(z), \quad z \in \mathbb{C}, \quad \mu > 0, \tag{4.1.3}$$

also with complex coefficients a_n. Actually, the series (4.1.2) is a particular case of (4.1.3) and can be obtained setting $m = 1$ in (4.1.3).

4.2 Cauchy–Hadamard type theorems

The following two statements give the domain of convergence of the series (4.1.1).

Theorem 4.2.1 (of Cauchy-Hadamard type)**.** *The domain of convergence of the series* (4.1.1) *is the circle domain* $|z| < R$ *with a radius of convergence*

$$R = (\limsup_{n\to\infty}(\ |a_n| \ / \ \Gamma(n+1) \ )^{1/n})^{-1}. \tag{4.2.1}$$

The cases $R = 0$ *and* $R = \infty$ *are incorporated in the common case.*

Corollary 4.2.1. *Let the series* (4.1.1) *converges at the point* $z_0 \neq 0$. *Then it is absolutely convergent in the disk* $D(0; |z_0|)$. *Inside the disk* $D(0; R)$*, i.e. on each closed disk* $|z| \leq r$ *(*$r < R$*), the convergence is uniform.*

Proof. Using the asymptotic formula (2.3.1) instead of (2.1.3), the proofs of Theorem 4.2.1 and Corollary 4.2.1 go in similar ways as the proofs of Theorem 3.4.1 and Corollary 3.4.1. We omit them. □

Further we only formulate the corresponding statements for series (4.1.3) by means of the functions $J^{\mu,m}_{n-2\lambda,\lambda}(z)$, given by (1.4.3). Their proofs use the same idea, but the asymptotic formula (2.3.5).

Theorem 4.2.2 (of Cauchy–Hadamard type [40])**.** *The domain of convergence of the series* (4.1.3) *is the disk* $D(0; R)$ *with a radius of* *convergence*

$$R = 2\left(\limsup_{n\to\infty}(\ |a_n|\ /\ \Gamma(n-\lambda+1)\ )^{1/n}\right)^{-1}. \qquad (4.2.2)$$

More precisely, the series (4.1.3) *is absolutely convergent in the disk* $D(0;R)$ *and divergent in the domain* $|z|>R$*. The cases* $R=0$ *and* $R=\infty$ *are incorporated in the common case.*

Corollary 4.2.2. *Let the series* (4.1.3) *converges at the point* $z_0 \neq 0$*. Then it is absolutely convergent in the disk* $D(0;|z_0|)$*. Inside the disk* $D(0;R)$*, i.e. on each closed disk* $|z|\le r$ *(*$r<R$*), the convergence is uniform.*

4.3 Abel type theorems

Let $z_0 \in \mathbb{C}$, $0<R<\infty$, $|z_0|=R$ and g_φ be an arbitrary angle domain with size $2\varphi<\pi$ and vertex at the point $z=z_0$, that is symmetric with respect to the line passing through the points 0 and z_0. The following theorem is valid.

Theorem 4.3.1 (of Abel type)**.** *Let* $\{a_n\}_{n=0}^{\infty}$ *be a sequence of complex numbers,* R *be defined by* (4.2.1) *and let* $K=\{|z|<R\}$*. If* $f(z)$ *is the sum of the series* (4.1.1) *on the domain* K *and this series is convergent at the point* z_0 *of the boundary of* K*, then* $\lim\limits_{z\to z_0} f(z) = \sum\limits_{n=0}^{\infty} a_n \widetilde{J}_n^{\mu}(z_0)$*,*

when $|z| < R$ *and* $z \in g_\varphi$, *i.e.*

$$\lim_{z\to z_0} f(z) = \sum_{n=0}^{\infty} a_n \widetilde{J}_n^{\mu}(z_0), \quad z \in g_\varphi. \tag{4.3.1}$$

Proof. Consider the difference

$$\Delta(z) = \sum_{n=0}^{\infty} a_n \widetilde{J}_n^{\mu}(z_0) - f(z) = \sum_{n=0}^{\infty} a_n (\widetilde{J}_n^{\mu}(z_0) - \widetilde{J}_n^{\mu}(z)), \tag{4.3.2}$$

representing it in the form

$$\Delta(z) = \sum_{n=0}^{k} a_n (\widetilde{J}_n^{\mu}(z_0) - \widetilde{J}_n^{\mu}(z)) + \sum_{n=k+1}^{\infty} a_n (\widetilde{J}_n^{\mu}(z_0) - \widetilde{J}_n^{\mu}(z)).$$

Let $p > 0$. By using the notations

$$\beta_m = \sum_{n=k+1}^{m} a_n \widetilde{J}_n^{\mu}(z_0), \quad m > k, \quad \beta_k = 0,$$

$$\gamma_n(z) = 1 - \widetilde{J}_n^{\mu}(z)/\widetilde{J}_n^{\mu}(z_0),$$

and the Abel transformation (see in [27, Vol.1, Ch.1, p.32, 3.4:7]), we obtain consequently:

$$\sum_{n=k+1}^{k+p} a_n (\widetilde{J}_n^{\mu}(z_0) - \widetilde{J}_n^{\mu}(z)) = \sum_{n=k+1}^{k+p} (\beta_n - \beta_{n-1})\gamma_n(z)$$
$$= \beta_{k+p}\gamma_{k+p}(z) - \sum_{n=k+1}^{k+p-1} \beta_n(\gamma_{n+1}(z) - \gamma_n(z)),$$

i.e.

$$\sum_{n=k+1}^{k+p} a_n(\widetilde{J}_n^\mu(z_0) - \widetilde{J}_n^\mu(z)) = (1 - \widetilde{J}_{k+p}^\mu(z)/\widetilde{J}_{k+p}^\mu(z_0)) \sum_{n=k+1}^{k+p} a_n \widetilde{J}_n^\mu(z_0)$$

$$- \sum_{n=k+1}^{k+p-1} (\sum_{s=k+1}^{n} a_s \widetilde{J}_s^\mu(z_0))(\frac{\widetilde{J}_n^\mu(z))}{\widetilde{J}_n^\mu(z_0))} - \frac{\widetilde{J}_{n+1}^\mu(z))}{\widetilde{J}_{n+1}^\mu(z_0))}).$$

From the asymptotic formula (2.3.1), it follows that there exists a natural number M such that $\widetilde{J}_n^\mu(z_0) \neq 0$ when $n > M$. Let $k > M$. Then, for every natural $n > k$:

$$\widetilde{J}_n^\mu(z)/\widetilde{J}_n^\mu(z_0) - \widetilde{J}_{n+1}^\mu(z)/J_{n+1}^\mu(z_0) = (z/z_0)^n \tag{4.3.3}$$

$$\times \frac{(1+\theta_n^\mu(z))(1+\theta_{n+1}^\mu(z_0)) - (z/z_0)(1+\theta_{n+1}^\mu(z))(1+\theta_n^\mu(z_0))}{(1+\theta_n^\mu(z_0))(1+\theta_{n+1}^\mu(z_0))}.$$

For the right hand side of (4.3.3) we apply Schvarz's lemma [27, v.1, p.317]. Then we get that there exists a constant C:

$$|\widetilde{J}_n^\mu(z)/\widetilde{J}_n^\mu(z_0) - \widetilde{J}_{n+1}^\mu(z)/J_{n+1}^\mu(z_0)| \le C|z - z_0||z/z_0|^n.$$

Analogously there exists a constant B:

$$|1 - \widetilde{J}_{k+p}^\mu(z)/\widetilde{J}_{k+p}^\mu(z_0)| \le B|z - z_0| \le 2B|z_0|.$$

Let ε be an arbitrary positive number and choose $N(\varepsilon)$ so large that for $k > N(\varepsilon)$ the inequality

$$|\sum_{s=k+1}^{n} a_s \widetilde{J}_s^\mu(z_0)| < \min(\varepsilon \cos\varphi/(12B|z_0|), \varepsilon\cos\varphi/(6C|z_0|))$$

holds for every natural $n > k$. Therefore, for $k > \max(M, N(\varepsilon))$:

$$|\sum_{s=k+1}^{\infty} a_s J_s^{\mu}(z_0)| \leq \min(\varepsilon \cos\varphi/(12B|z_0|), \varepsilon \cos\varphi/(6C|z_0|),$$

and

$$|\sum_{n=k+1}^{\infty} a_n(\widetilde{J}_n^{\mu}(z_0) - \widetilde{J}_n^{\mu}(z))| \leq (\varepsilon \cos\varphi/6)(1 + \sum_{n=k+1}^{\infty} |z_0|^{-1}|z - z_0||z/z_0|^n)$$

$$\leq (\varepsilon \cos\varphi/6)(1 + |z - z_0|/(|z_0| - |z|)).$$

But near the vertex of the angle domain g_φ in the part d_φ closed between the angle's arms and the arc of the circle with center at the point 0 and touching the arms of the angle we have $|z-z_0|/(|z_0|-|z|) < 2/\cos\varphi$, i.e. $|z - z_0|\cos\varphi < 2(|z_0| - |z|)$. That is why the inequality

$$|\sum_{n=k+1}^{\infty} a_n(\widetilde{J}_n^{\mu}(z_0) - \widetilde{J}_n^{\mu}(z))| < (\varepsilon \cos\varphi)/6 + \varepsilon/3 \leq \varepsilon/2 \qquad (4.3.4)$$

holds for $z \in d_\varphi$ and $k > \max(M, N(\varepsilon))$. Fix some $k > \max(M, N(\varepsilon))$ and after that choose $\delta(\varepsilon)$ such that if $|z-z_0| < \delta(\varepsilon)$ then the inequality

$$|\sum_{n=0}^{k} a_n(\widetilde{J}_n^{\mu}(z_0) - \widetilde{J}_n^{\mu}(z))| < \varepsilon/2 \qquad (4.3.5)$$

holds inside d_φ. We get

$$|\Delta(z)| = |\sum_{n=0}^{\infty} a_n(\widetilde{J}_n^{\mu}(z_0) - \widetilde{J}_n^{\mu}(z))|$$

for the module of the difference (4.3.2). In view of (4.3.4) and (4.3.5), the equality (4.3.1) holds. $\square$

Analogical result, which proof goes analogously, can be formulated for the series (4.1.3). Namely, the following theorem holds true.

Theorem 4.3.2 (of Abel type, [38]). *Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of complex numbers, R be defined by* (4.2.2) *and let $K = \{|z| < R\}$. If $f(z)$ is the sum of the series* (4.1.3) *on the domain K and this series is convergent at the point z_0 of the boundary of K, then $\lim\limits_{z\to z_0} f(z) = \sum\limits_{n=0}^{\infty} a_n J_{n-2\lambda,\lambda}^{\mu,m}(z_0)$, when $|z| < R$ and $z \in g_\varphi$, i.e.*

$$\lim_{z\to z_0} f(z) = \sum_{n=0}^{\infty} a_n J_{n-2\lambda,\lambda}^{\mu,m}(z_0), \quad z \in g_\varphi. \tag{4.3.6}$$

In particular taking $m = 1$, the Abel type theorem for the series (4.1.2) follows.

4.4 Tauber type theorems

Beginning this Section, we make the following important remark.

Remark 4.4.1. *According to Theorems 2.3.1 and 2.3.2, it follows that there exists a natural number M such that the functions $J_n^{\mu}(z)$, $J_{n-2\lambda,\lambda}^{\mu,m}(z)$, and $J_{n-2\lambda,\lambda}^{\mu}(z)$ have no zeros for $n > M$, possibly except for the zero.*

Now, et us consider the series $\sum\limits_{n=0}^{\infty} a_n$, $a_n \in \mathbb{C}$. Let

$$z_0 \in \mathbb{C}, \quad |z_0| = R, \quad 0 < R < \infty, \quad J_n^{\mu}(z_0) \neq 0 \quad \text{for } n = 0, 1, 2, \ldots .$$

For convenience, denote

$$J^*_{n,\mu}(z;z_0) = \frac{\widetilde{J}^\mu_n(z)}{\widetilde{J}^\mu_n(z_0)}. \tag{4.4.1}$$

Let the series $\sum\limits_{n=0}^{\infty} a_n J^*_{n,\mu}(z;z_0)$ be convergent for $|z| < R$ and

$$F(z) = \sum_{n=0}^{\infty} a_n J^*_{n,\mu}(z;z_0), \quad |z| < R. \tag{4.4.2}$$

Theorem 4.4.1 (of Tauber type)**.** *If* $\{a_n\}_{n=0}^{\infty}$ *be a sequence of complex numbers with*

$$\lim_{n\to\infty} na_n = 0, \tag{4.4.3}$$

and there exists

$$\lim_{z\to z_0} F(z) = S \quad (|z| < R, z \to z_0 \textit{ radially}),$$

then the series $\sum\limits_{n=0}^{\infty} a_n$ *is convergent and*

$$\sum_{n=0}^{\infty} a_n = S.$$

Proof. For a point z of the segment $[0, z_0]$ we have

$$\sum_{n=0}^{k} a_n - F(z) = \sum_{n=0}^{k} a_n - \sum_{n=0}^{\infty} a_n J^*_{n,\mu}(z;z_0)$$

$$= \sum_{n=0}^{k} a_n \frac{\widetilde{J}_n^{\mu}(z_0)}{\widetilde{J}_n^{\mu}(z_0)} - \sum_{n=0}^{\infty} a_n \frac{\widetilde{J}_n^{\mu}(z)}{\widetilde{J}_n^{\mu}(z_0)}$$

$$= \sum_{n=0}^{k} a_n \frac{\widetilde{J}_n^{\mu}(z_0) - \widetilde{J}_n^{\mu}(z)}{\widetilde{J}_n^{\mu}(z_0)} - \sum_{n=k+1}^{\infty} a_n J_{n,\mu}^{*}(z; z_0)$$

and therefore,

$$|\sum_{n=0}^{k} a_n - F(z)| \le \sum_{n=0}^{k} |a_n| \left| \frac{\widetilde{J}_n^{\mu}(z_0) - \widetilde{J}_n^{\mu}(z)}{\widetilde{J}_n^{\mu}(z_0)} \right| + \sum_{n=k+1}^{\infty} |a_n| \left| J_{n,\mu}^{*}(z; z_0) \right| . \tag{4.4.4}$$

By using the asymptotic formula (2.3.1), for the Bessel–Maitland functions, we obtain:

$$a_n J_{n,\mu}^{*}(z; z_0) = a_n \left(\frac{z}{z_0}\right)^n \frac{1 + \theta_n^{\mu}(z)}{1 + \theta_n^{\mu}(z_0)} = a_n \left(\frac{z}{z_0}\right)^n \left(1 + \widetilde{\theta}_{n,\mu}(z; z_0)\right).$$

Let ε be an arbitrary positive number. We choose a number N_1 so large that the inequalities $|1 + \widetilde{\theta}_{k,\mu}(z; z_0)| < 2, |ka_k| < \frac{\varepsilon}{6}$ hold for $k > N_1$. If $k > N_1$ and z is on the segment $[0, z_0]$, then for the second summand in (4.4.4) the following estimate is valid:

$$\sum_{n=k+1}^{\infty} |a_n| \left| J_{n,\mu}^{*}(z; z_0) \right| = \sum_{n=k+1}^{\infty} |a_n| \left| \frac{z}{z_0} \right|^n |1 + \widetilde{\theta}_{n,\mu}(z; z_0)| \tag{4.4.5}$$

$$\le 2 \left| \frac{z}{z_0} \right|^{k+1} \sum_{n=k+1}^{\infty} |a_n| \left| \frac{z}{z_0} \right|^{n-k-1} \le 2 \sum_{n=0}^{\infty} |a_{n+k+1}| \left| \frac{z}{z_0} \right|^n$$

$$= 2\sum_{n=0}^{\infty} \frac{|(n+k+1)a_{n+k+1}|}{n+k+1} \left|\frac{z}{z_0}\right|^n < 2\sum_{n=0}^{\infty} \frac{\varepsilon/6}{n+k+1} \left|\frac{z}{z_0}\right|^n$$

$$< \frac{2}{k}\,\frac{\varepsilon}{6}\,\frac{1}{1-|z/z_0|} = \frac{\varepsilon}{3}\,\frac{1}{k}\,\frac{|z_0|}{|z_0|-|z|}\ .$$

Now let us consider the first summand in (4.4.4). We have:

$$\sum_{n=0}^{k} |a_n| \left|\frac{\widetilde{J}_n^{\mu}(z_0) - \widetilde{J}_n^{\mu}(z)}{\widetilde{J}_n^{\mu}(z_0)}\right|$$

$$= \sum_{n=0}^{m} |a_n| \left|\frac{\widetilde{J}_n^{\mu}(z_0) - \widetilde{J}_n^{\mu}(z)}{\widetilde{J}_n^{\mu}(z_0)}\right| + \sum_{n=m+1}^{k} |a_n| \left|\frac{\widetilde{J}_n^{\mu}(z_0) - \widetilde{J}_n^{\mu}(z)}{\widetilde{J}_n^{\mu}(z_0)}\right|.$$

According to Schwarz's lemma, there exists a constant C such that

$$\left|\frac{\widetilde{J}_n^{\mu}(z_0) - \widetilde{J}_n^{\mu}(z)}{\widetilde{J}_n^{\mu}(z_0)}\right| < C|z-z_0|.$$

Moreover, there exists a number N_2 such that the following inequality

$$\sum_{n=0}^{m} |a_n| \left|\frac{\widetilde{J}_n^{\mu}(z_0) - \widetilde{J}_n^{\mu}(z)}{\widetilde{J}_n^{\mu}(z_0)}\right| \le C\,|z-z_0|\,k\,\frac{\sum_{n=0}^{m}|a_n|}{k} \tag{4.4.6}$$

$$< C\,|z-z_0|\,k\,\frac{\varepsilon}{3RC} = |z-z_0|\,k\,\frac{\varepsilon}{3R}.$$

holds as $k > N_2$. It remains to estimate the sum

$$\sum_{n=m+1}^{k} |a_n| \left|\frac{\widetilde{J}_n^{\mu}(z_0) - \widetilde{J}_n^{\mu}(z)}{\widetilde{J}_n^{\mu}(z_0)}\right|.$$

To this end, using asymptotic formula (2.3.1) for the Bessel–Maitland functions, we find consequently:

$$\frac{\widetilde{J}_n^\mu(z_0)-\widetilde{J}_n^\mu(z)}{\widetilde{J}_n^\mu(z_0)}=\frac{(z_0)^n(1+\theta_n^\mu(z_0))-z^n(1+\theta_n^\mu(z))}{z_0^n(1+\theta_n^\mu(z_0))}=1-\left(\frac{z}{z_0}\right)^n\frac{1+\theta_n^\mu(z)}{1+\theta_n^\mu(z_0)}$$

$$=1-\left(\frac{z}{z_0}\right)^n\left[1+\frac{\theta_n^\mu(z)-\theta_n^\mu(z_0)}{1+\theta_n^\mu(z_0)}\right]=1-\left(\frac{z}{z_0}\right)^n-\left(\frac{z}{z_0}\right)^n\frac{\theta_n^\mu(z)-\theta_n^\mu(z_0)}{1+\theta_n^\mu(z_0)}\,.$$

Therefore,

$$\left|\frac{\widetilde{J}_n^\mu(z_0)-\widetilde{J}_n^\mu(z)}{\widetilde{J}_n^\mu(z_0)}\right|\leq\left|1-\left(\frac{z}{z_0}\right)^n\right|+\left|\frac{z}{z_0}\right|^n\left|\frac{\theta_n^\mu(z)-\theta_n^\mu(z_0)}{1+\theta_n^\mu(z_0)}\right|. \qquad (4.4.7)$$

We obtain the following inequalities

$$\left|1-\left(\frac{z}{z_0}\right)^n\right|=\left|1-\frac{z}{z_0}\right|\left|1+\frac{z}{z_0}+\left(\frac{z}{z_0}\right)^2+\cdots+\left(\frac{z}{z_0}\right)^{n-1}\right|\leq n\left|1-\frac{z}{z_0}\right|$$

for the first summand of (4.4.7). According to Schwarz's lemma, there exists a constant ρ such that

$$\left|\frac{\theta_n^\mu(z)-\theta_n^\mu(z_0)}{1+\theta_n^\mu(z_0)}\right|\leq 1 \qquad \text{as} \qquad |z-z_0|<\rho.$$

Then, for such $|z|$, we obtain for the second summand of (4.4.8):

$$\left|\frac{z}{z_0}\right|^n\left|\frac{\theta_n^\mu(z)-\theta_n^\mu(z_0)}{1+\theta_n^\mu(z_0)}\right|\leq\left|\frac{z}{z_0}\right|^n|z-z_0|.$$

From (4.4.11) it follows that

$$\lim_{n\to\infty}a_n=0,\quad \lim_{k\to\infty}\frac{\sum\limits_{n=1}^{k}n|a_n|}{k}=0,\quad \lim_{k\to\infty}\frac{\sum\limits_{n=1}^{k}|a_n|}{k}=0.$$

Then a number N_3 exists such that

$$\frac{\sum\limits_{n=m+1}^{k} n|a_n|}{k} < \frac{\varepsilon}{3(1+R)} \quad \text{and} \quad \frac{\sum\limits_{n=m+1}^{k} |a_n|}{k} < \frac{\varepsilon}{3(1+R)} \quad \text{as} \quad k > N_3.$$

Therefore,

$$\sum_{n=m+1}^{k} |a_n| \left| \frac{\widetilde{J}_n^\mu(z_0) - \widetilde{J}_n^\mu(z)}{\widetilde{J}_n^\mu(z_0)} \right| \leq \sum_{n=m+1}^{k} n|a_n| \left| 1 - \frac{z}{z_0} \right| \tag{4.4.8}$$

$$+ \sum_{n=m+1}^{k} |a_n| \left| \frac{z}{z_0} \right|^n |z - z_0| \leq k\, \frac{|z-z_0|}{R}\, \frac{\sum\limits_{n=m+1}^{k} n|a_n|}{k} + k\,|z-z_0|\, \frac{\sum\limits_{n=m+1}^{k} |a_n|}{k}$$

$$< k\,|z-z_0|\, \frac{1+R}{R}\, \frac{\varepsilon}{3(1+R)} = k\,|z-z_0|\, \frac{\varepsilon}{3R}.$$

Finally, let us note that

$$\left| \sum_{n=0}^{k} a_n - F(z) \right| \leq \sum_{n=0}^{m} |a_n| \left| \frac{\widetilde{J}_n^\mu(z_0) - \widetilde{J}_n^\mu(z)}{\widetilde{J}_n^\mu(z_0)} \right|$$

$$+ \sum_{n=m+1}^{k} |a_n| \left| \frac{\widetilde{J}_n^\mu(z_0) - \widetilde{J}_n^\mu(z)}{\widetilde{J}_n^\mu(z_0)} \right| + \sum_{n=k+1}^{\infty} |a_n| \left| J_{n,\mu}^*(z; z_0) \right| .$$

Let $N = \max(N_1, N_2, N_3), k > N$ and $|z - z_0| < \rho$. Then by using (4.4.5), (4.4.6) and (4.4.8), we can conclude that

$$\left| \sum_{n=0}^{k} a_n - F(z) \right| < |z-z_0|\, k\, \frac{\varepsilon}{3R} + k\,|z-z_0|\, \frac{\varepsilon}{3R} + \frac{\varepsilon}{3}\, \frac{1}{k}\, \frac{|z_0|}{|z_0| - |z|}$$

$$= \frac{\varepsilon}{3}\left[\frac{2k}{R}\,|z-z_0| + \frac{1}{k}\,\frac{|z_0|}{|z_0|-|z|}\right].$$

If we substitute z by $z_0(1-\frac{1}{k})$, then

$$\left|\sum_{n=0}^{k} a_n - F\left(z_0(1-\frac{1}{k})\right)\right| < \frac{\varepsilon}{3}3 = \varepsilon.$$

This proves that $\lim\limits_{k\to\infty}\sum\limits_{n=0}^{k} a_n$ exists and equals $\lim\limits_{k\to\infty} F\left(z_0(1-\frac{1}{k})\right)$, i.e.

$$\sum_{n=0}^{\infty} a_n = \lim_{k\to\infty} F\left(z_0(1-\frac{1}{k})\right) = S.$$

Thus the theorem is proved. $\square$

Remark 4.4.2. *In the particular case $\mu = 1$ of the above considerations we obtain the results published in [36] for series $\sum_{n=0}^{\infty} a_n J_n(z)$ in terms of Bessel functions $J_\nu(z) = (z/2)^\nu J_\nu^1(z^2/4)$.*

The result related to the generalized Lommel–Wright functions (1.4.3) is only formulated, because their proofs goes in the same way, but using specifics of the corresponding asymptotic formulae.

Note that each function $J^{\mu,m}_{n-2\lambda,\lambda}(z)$, $n \in \mathbb{N}_0$, being an entire function, not identically zero, has at most a finite number of zeros in the closed and bounded set $D(0;R)$. Moreover, due to Remark 4.4.1, only finite number of these functions may have some zeros at all, possibly except for 0.

Let $z_0 \in \mathbb{C}$, $|z_0| = R$, $0 < R < \infty$, and $J^{\mu,m}_{n-2\lambda,\lambda}(z_0) \neq 0$ for $n = 0, 1, 2, \ldots$. For the sake of brevity, we denote

$$J^*_{n,\lambda,\mu,m}(z;z_0) = \frac{J^{\mu,m}_{n-2\lambda,\lambda}(z)}{J^{\mu,m}_{n-2\lambda,\lambda}(z_0)}. \tag{4.4.9}$$

Let the series $\sum\limits_{n=0}^{\infty} a_n J^*_{n,\lambda,\mu,m}(z;z_0)$ be convergent for $|z| < R$ and

$$F(z) = \sum_{n=0}^{\infty} a_n J^*_{n,\lambda,\mu,m}(z;z_0), \quad |z| < R. \tag{4.4.10}$$

Theorem 4.4.2 (of Tauber type, [38]). *If $F(z)$ is given by* (4.4.10), $\{a_n\}_{n=0}^{\infty}$ *is a sequence of complex numbers with*

$$\lim_{n\to\infty} n a_n = 0, \tag{4.4.11}$$

and there exists $\lim\limits_{z\to z_0} F(z) = S \quad (|z| < R, z \to z_0$ *radially*), *then the series* $\sum\limits_{n=0}^{\infty} a_n$ *is convergent and* $\sum\limits_{n=0}^{\infty} a_n = S$.

Finally, the corresponding result for the series by means of functions (1.4.3) follow by Theorem 4.4.2, taking $m = 1$, and this case is considered in [39].

In conclusion to note that the results obtained in both Chapters 3 and 4 are completely analogical to the known classical ones for the power series.

5 Zeros of Finite Hankel Transforms

5.1 Problem of Zeros Distribution of a Class of Entire Functions of an Exponential Kind

The problem of zero distribution of entire functions of the type

$$\int_{-\infty}^{+\infty} F(t)\exp(izt)\,dt, \tag{5.1.1}$$

was posed by the famous Danish mathematician and engineer Johan Ludwig William Valdemar Jensen, mostly known as Johan Jensen [10]. The reason for the consideration of Jensen's work is the fact, as Riemann noticed [49], that the distribution, respectively the asymptotic behavior of the zeros of the entire functions of the type (5.1.1) is closely related to important problems of the analytical theory of numbers.

Particular case of the entire functions of the type (5.1.1) are the functions

In the Pólya work, which is mainly devoted to

$$\int_{-a}^{a} f(t)\exp(izt)\,dt\ , \quad 0 < a < \infty . \tag{5.1.2}$$

As the first Pólya studies show, the problem for the distribution of zeros of the entire functions (5.1.1), respectively (5.1.2), differ significantly. This difference is primarily due to the fact that the integral (5.1.1) presents at all an entire function with an order greater than one, whereas (5.1.2) represents an entire function of an exponential type, i.e. an entire function of an order, less than or equal to one, and a normal type. Accordingly, the methods for studying the entire functions (5.1.1) and (5.1.2) are different.

In the Pólya work [43], which is mainly devoted to the entire functions of the type

$$U(f;z) = \int_{0}^{1} f(t)\cos(zt)\,dt \tag{5.1.3}$$

and

$$V(f;z) = \int_{0}^{1} f(t)\sin(zt)\,dt\,, \tag{5.1.4}$$

which are a particular case of (5.1.2), a method is proposed for studying the distribution of their zeros, which can be called a method of integral sums. This method is based mostly on the relationship that exists between the distribution of the zeros of the entire functions (5.1.3), respectively (5.1.4), and the distribution of the zeros of the polynomials

$$P_n(f;z) = \sum_{k=0}^{n} f(k/n) z^k . \qquad (5.1.5)$$

In [43] Pólya also gets statements about the mutual distribution of the zeros of the entire functions (5.1.3), respectively (5.1.4), and the entire functions $\sin z$ and $\cos z$, based on a result of Hurwitz. Hurwitz's idea is to look at the meromorphic function $U(f;z)/\sin z$ or $U(f;z)/\cos z$ instead of the entire function $U(f;z)$. Namely, the representation of this meromorphic function as the sum of the respective partial fractions is considered.

Pólya's work gives impulse to numerous research and publications in this direction, mainly by Bulgarian authors. Significant contributions are given by academics L. Tchakalov, N. Obrechkoff, L. Iliev. Their research finds continuation in the works of E. Bojorov, K. Dochev, P. Rusev, D. Dimitrov and others.

This Section examines the asymptotic behavior of the zeros of the entire functions of types (5.1.3) and (5.1.4).

It turns out that in some assumptions about $f(t)$ the zeros of the entire functions (5.1.3) and (5.1.4) approach the zeros of $\sin z$ and $\cos z$ respectively. The order of this approximation is determined by the order of magnitude

$$U(f;n\pi),\ U'(f;n\pi) \text{ u } V(f;(n+1/2)\pi),\ V'(f;(n+1/2)\pi)$$

as functions of n.

Similar problems were also considered by Rusev [50], [51], and Kassandrova [11].

An important role in the further exposition is played by a theorem of Hurwitz [43] for $U(f;z)$ and its analogue for $V(f;z)$, which states the following.

Theorem 5.1.1 (of Hurwitz). *If the real function* $f(t)$, *defined and integrable in absolute value in a Riemann sense in the interval* [0, 1], *for every integer value of p satisfies the inequality*

$$U(f;n\pi)\, U\big(f;(n+1)\pi\big) < 0,$$

respectively

$$V\big(f;(n-1/2)\pi\big)\, V\big(f;(n+1/2)\pi\big) < 0,$$

then the entire function $U(f;z)$ *(pecn.* $V(f;z)$*) has only real and simple zeros which are so distributed that each of the intervals*

$$\ldots,(-3\pi,-2\pi),(-2\pi,-\pi),(-\pi,0),(0,\pi),(\pi,2\pi),(2\pi,3\pi),\ \ldots,$$

respectively

$$\ldots,(-3\pi/2,-\pi/2),(-\pi/2,\pi/2),(\pi/2,\ 3\pi/2),\ \ldots,$$

contains exactly one zero.

5.2 Auxiliary Statements

Lemma 5.2.1. [32] *Let* $f(t)$ *be a real-valued function, defined in the interval* [0, 1], *two times differentiable and* $f''(t)$ *be absolutely integrable in the Riemann sense. If the conditions*

$$|f'(1)| > |f'(0)|, \quad f(1) \neq 0,$$

are fulfilled, then there exist a natural number N and positive constants C_0 , C_1 , C_2*, such that for every integer number n and real x, for which* $|n| > N$, $|x| > N$*, the following relations hold true*:

$$\left|U(f;n\pi)\right| \leq C_0 / n^2, \ \left|U'(f;n\pi)\right| \geq C_1/|n|, \ \left|U''(f;x)\right| \leq C_2/|x| , \qquad (5.2.1)$$

$$\mathrm{sign}\left(U(f;n\pi)U'(f;n\pi)\right) = \mathrm{sign}\left(nf(1)f'(1)\right). \qquad (5.2.2)$$

Proof. Let n be an integer number and x be a real number. Integrating by part, we consequently obtain the following equalities

$$U(f;n\pi) = \frac{1}{(n\pi)^2}\left((-1)^n f'(1) - f'(0)\right) - \frac{1}{(n\pi)^2}\int_0^1 f''(t)\cos(n\pi t)\,dt ,$$

$$U'(f;n\pi) = (-1)^n \frac{f(1)}{n\pi} - \frac{1}{n\pi}\int_0^1 \left(f(t) + tf'(t)\right)\cos(n\pi t)\,dt ,$$

$$U''(f;x) = -f(1)\frac{\sin x}{x} + \frac{1}{x}\int_0^1 \left(2tf(t) + t^2 f'(t)\right)\sin(xt)\,dt .$$

Further on, since

$$\lim_{x\to\infty}\int_0^1 \left(2tf(t) + t^2 f'(t)\right)\sin(xt)\,dt = 0,$$

$$\lim_{n\to\infty}\int_0^1 f''(t)\cos(n\pi t)\,dt = 0$$

$$\lim_{n\to\infty} \int_0^1 \left(f(t) + t f'(t) \right) \cos\left(n\pi t \right) dt = 0,$$

the conclusion of the lemma follows immediately. Analogously can be formulated and proved such type of lemma for $V(f;z)$. The result is given bellow

Lemma 5.2.2. [32] *Let* $f(t)$ *be a real-valued function, defined in the interval* [0, 1]*, two times differentiable and* $f''(t)$ *be absolutely integrable in the Riemann sense. If the conditions*

$$f(1)f'(1) \neq 0, \quad f(0) = 0,$$

fulfilled, then there exist a natural number N and positive constants C_0 , C_1 , C_2*, such that for every integer number n and real x, for which* $|n| > N$, $|x| > N$ *, the following relations hold true*:

$$\left| V\left(f;\left(n+1/2\right)\pi\right) \right| \leq C_0 \big/ n^2 , \quad \left| V'\left(f;\left(n+1/2\right)\pi\right) \right| \geq C_1 \big/ |n| ,$$

$$\left| V''\left(f;x\right) \right| \leq C_2 \big/ |x| ,$$

$$\operatorname{sign}\left(V\left(f;\left(n+1/2\right)\pi\right) V'\left(f;\left(n+1/2\right)\pi\right) \right) = \operatorname{sign}\left(n f(1) f'(1) \right).$$

5.3 Asymptotic Behaviour of the Zeros

Letting *n* an integer number, we denote with x_n the zero of (5.1.3), which is in the interval $\left((n-1)\pi,\ n\pi \right)$.

Theorem 5.3.1 [32] *Let* $f(t)$ *be a function, satisfying the conditions of the Hurwitz theorem* [*Section 5.1*]. *Let additionally* *a)* $f(t)$ be *two times differentiable in the interval* [0, 1];

б) $|f''(t)|$ *be integrable in the Riemann sense*;

в) $|f'(1)| > |f'(0)|$;

г) $f(1) \neq 0$.

Then there is a natural number N and a constant C>0, such that the following inequalities

$$|x_n - n\pi| < C/|n|, \text{ if } \quad U(f;n\pi)U'(f;n\pi) > 0,$$

and

$$|x_{n+1} - n\pi| < C/|n|, \text{ if } U(f;n\pi)\,U'(f;n\pi) < 0,$$

hold true, for every integer n with $|n| > N$.

Proof. Let $f(t)$ fulfills the conditions of the theorem and hence the conditions of the Hurwitz theorem. Then the function defined by equation (5.1.3) has only real and simple zeros, distributed so that each of the intervals

$$\ldots,(-3\pi,-2\pi),(-2\pi,-\pi),(-\pi,0),(0,\pi),(\pi,2\pi),(2\pi,3\pi),\ \ldots$$

contains exactly one zero. According to Lemma 5.2.1 there is a natural number N_1 and constants C_0, C_1, C_2, such that inequalities (5.2.1) are fulfilled, provided only $|n| > N_1$, $|x| > N_1$. Let us denote

$$N = \max\left(N_1,\ (9C_0C_2)\big/\left(4C_1^2\right)+1,\ (3C_0)/(2C_1\pi)\right)$$

and let $|n+1| > N$, $\quad |n-1| > N$, $\quad x \in \left((n-1)\pi,\ (n+1)\pi\right)$.

Further, the following equality holds true for $U(f;z)$:

$$U(f;x) = U(f;n\pi) + U'(f;n\pi)(x-n\pi) + \frac{U''(f;\xi)}{2}(x-n\pi)^2, \quad (5.3.1)$$

for $\xi \in ((n-1)\pi, (n+1)\pi)$. Depending on the sign of the product $U(f;n\pi)U'(f;n\pi)$, using the equality (5.2.2), a few cases can be considered.

(I). Letting $U(f;n\pi)U'(f;n\pi) > 0$, we define

$$x_0(n) = n\pi - \frac{3}{2}\cdot\frac{C_0}{C_1}\cdot\frac{1}{|n|}.$$

(I.1). Let $U(f;n\pi) > 0$, $U'(f;n\pi) > 0$ first. Because of (5.2.1) and (5.3.1), we obtain consequently

$$U(f;n\pi) \le C_0/n^2, \quad -U'(f;n\pi) \le -C_1/|n|, \quad |U''(f;\xi)| \le C_2/|\xi|,$$

$$U(f;x_0) = U(f;n\pi) - U'(f;n\pi)\frac{3}{2}\cdot\frac{C_0}{C_1}\cdot\frac{1}{|n|} + \frac{U''(f;\xi)}{2}\cdot\left(\frac{3C_0}{2C_1}\right)^2\cdot\frac{1}{n^2},$$

from where the inequalities sequence follows:

$$U(f;x_0) \le -\frac{1}{2}\cdot\frac{C_0}{n^2} + \frac{U''(f;\xi)}{2}\cdot\left(\frac{3C_0}{2C_1}\right)^2\cdot\frac{1}{n^2} < 0. \quad (5.3.2)$$

(I.2). За $U(f;n\pi) < 0$, $U'(f;n\pi) < 0$, due to the obvious inequalities

$$U(f;n\pi) \ge -C_0/n^2, \; -U'(f;n\pi) \ge -C_1/|n|,$$

we obtain

$$U(f;x_0) \ge -\frac{1}{2}\cdot\frac{C_0}{n^2} + \frac{U''(f;\xi)}{2}\cdot\left(\frac{3C_0}{2C_1}\right)^2\cdot\frac{1}{n^2} > 0. \quad (5.3.3)$$

From (5.3.2) and (5.3.3) we can conclude that

$$U(f;x_0)U(f;n\pi)<0.$$

Therefore, if $U(f;n\pi)U'(f;n\pi)<0$, the zero x_n of the function $U(f;z)$ is in the interval $(x_0(n),\ n\pi)$. Therefore, there is a positive constant C, for which inequality $|\ x_n-n\pi\ |<C/|n|$ holds true.

(II). Let $U(f;n\pi)U'(f;n\pi)<0$ and let us define

$$x_0(n)=n\pi+\frac{3}{2}\cdot\frac{C_0}{C_1}\cdot\frac{1}{|n|}.$$

(II.1). Letting $U(f;n\pi)<0,\ U'(f;n\pi)>0$, we consider $U(f;x_0)$. Additionally, bearing in mind the inequalities (II12.6), we consequently obtain:

$$U(f;x_0)=U(f;n\pi)+U'(f;n\pi)\frac{3}{2}\cdot\frac{C_0}{C_1}\cdot\frac{1}{|n|}+\frac{U''(f;\xi)}{2}\cdot\left(\frac{3C_0}{2C_1}\right)^2\cdot\frac{1}{n^2},$$

$$U(f;x_0)\geq\frac{1}{2}\cdot\frac{C_0}{n^2}+\frac{U''(f;\xi)}{2}\cdot\left(\frac{3C_0}{2C_1}\right)^2\cdot\frac{1}{n^2}>0. \tag{5.3.4}$$

(II.2). If $U(f;n\pi)>0,\ U'(f;n\pi)<0$, we have

$$U(f;x_0)\leq-\frac{1}{2}\cdot\frac{C_0}{n^2}+\frac{U''(f;\xi)}{2}\cdot\left(\frac{3C_0}{2C_1}\right)^2\cdot\frac{1}{n^2}<0. \tag{5.3.5}$$

From (5.3.4) and (5.3.5) we conclude that $U(f;x_0)U(f;n\pi)<0$. Therefore the zero x_{n+1} of the function $U(f;z)$ lies in the interval $(n\pi,\ x_0(n))$, i. e. there exists a positive constant C, for which $|\ x_{n+1}-n\pi\ |<C/|n|$, which proves Theorem 5.3.1. □

Example 5.3.1. Let us consider the function $f(t)=t^2$. We obtain the following equality for $U(f;n\pi)$:

$$U\left(t^2;n\pi\right)=2(-1)^n/(n\pi)^2.$$

Therefore the inequalities

$$U\left(t^2;n\pi\right)U\left(t^2;(n+1)\pi\right)<0$$

hold true for every integer *n*. The other conditions of Theorem 5.3.1 are also met. □

Let us consider the function (5.1.4) and to denote with $\tilde{x}_n$ this zero of (5.1.4), which is in the interval $((n-1/2)\pi,(n+1/2)\pi)$. Then the following theorem can be formulated.

Theorem 5.3.2. [32] *Let* $f(t)$ *be a function, satisfying the conditions of the Hurwitz theorem* (*Section 5.1*). *Let additionally*

а) $f(t)$ be *two times differentiable in the interval* [0, 1];

б) $|f''(t)|$ *be integrable in the Riemann sense*;

в) $f(1)\,f'(1)\neq 0$;

г) $f(0)=0$;

Then there exist a natural number N and a constant C>0, such that the following inequalities

$$|\tilde{x}_n-(n+1/2)\pi|<C/|n|,\ \text{if}\ V\left(f;(n+1/2)\pi\right)V'\left(f;(n+1/2)\pi\right)>0,$$

and

$$|\,\tilde{x}_{n+1} - (n+1/2)\pi\,| < C/|n|, \ \ if \ \ V\big(f;(n+1/2)\pi\big)V'\big(f;(n+1/2)\pi\big) < 0,$$

hold true, for every integer n *with* $|\,n\,| > N$.

5.4 Generalizations

Using the above method, generalizations of obtained theorems in Section 5.3 can be made.

Theorem 5.4.1. [32] *Let* $f(t)$ *be a function, satisfying the conditions of the Hurwitz theorem* (*Section 5.1*). *Let there exist a natural number N and positive constants* $C_0, C_1, C_2, \alpha, \beta$, *such that the following inequalities hold true*

$$\big|U(f;n\pi)\big| \le C_0 \big/ |n|^{\alpha+\beta}, \ \ \big|U'(f;n\pi)\big| \ge C_1/|n|^{\alpha}, \ \ \big|U''(f;x)\big| \le C_2/|x|^{\alpha},$$

for every integer n and real x, with $|n| > N$ *and* $|x| > N$.

Then there exists a positive constant C, such that

$$|\,x_n - n\pi\,| < C/|n|^{\beta}, \qquad if \qquad U(f;n\pi)U'(f;n\pi) > 0$$

and

$$|\,x_{n+1} - n\pi\,| < C/|n|^{\beta}, \quad if \qquad U(f;n\pi)U'(f;n\pi) < 0.$$

Theorem 5.4.2. [32] *Let* $f(t)$ *be a function, satisfying the conditions of the Hurwitz theorem* (*in Section 5.1*). *Let there exist a natural number N and positive constants* $C_0, C_1, C_2, \alpha, \beta$, *such that the following inequalities hold true*

$$\left|V\left(f;\left(n+1/2\right)\pi\right)\right| \le C_0 \Big/ |n|^{\alpha+\beta}, \ \left|V'\left(f;\left(n+1/2\right)\pi\right)\right| \ge C_1 \big/ |n|^{\alpha},$$

$$\left|V''\left(f;x\right)\right| \le C_2 \big/ |x|^{\alpha}.$$

for every integer n and real x, with $|n| > N$ *and* $|x| > N$.

Then there exists a positive constant C, such that

$$\left|\widetilde{x}_n - \left(n+1/2\right)\pi\right| < C/|n|^{\beta}, \ \text{if} \ \ V\left(f;\left(n+1/2\right)\pi\right)V'\left(f;\left(n+1/2\right)\pi\right) > 0$$

and

$$\left|\widetilde{x}_{n+1} - \left(n+1/2\right)\pi\right| < C/|n|^{\beta}, \ \text{if} \ \ V\left(f;\left(n+1/2\right)\pi\right)V'\left(f;\left(n+1/2\right)\pi\right) < 0.$$

5.5 The Distribution of the Zeros of a Class of Entire Functions Involving Bessel Functions in the Kernel of its Integral Representation

Let $J_\nu(z)$ be the Bessel of the first kind with an index $\nu > -1$. It is well known that the function $J_\nu(z)$ is represented in the form $J_\nu(z) = z^\nu U_\nu(z)$ in the domain $\mathbb{C}\setminus(-\infty,0)$ where $U_\nu(z)$ is an entire and an even function. It is also known that $U_\nu(z)$ have infinite number zeros and all of them are real. The distribution of the zeros of the entire functions

$$A_\nu(f;z) = \int_0^1 f(t)\,t^\nu U_\nu(zt)\,dt \tag{5.5.1}$$

is investigated in this section. It is proved that under certain conditions of very general character, imposed on the function f, the entire function (5.5.1) has not more than finite number non-real zeros.

Similar problems concerning the zeros of entire functions, more particular, than (5.5.1), have been considered by Polya [43], Tchakalov [63], Obrechkoff [28], P. Rusev [51] and Kassandrova [11]. Further on, the following auxiliary statements are used.

5.6 Auxiliary Statements

Lemma 5.6.1. [33] *Let the following two infinite sequences of numbers be given*

(α): $\alpha_1, \alpha_2, \ldots, \alpha_n, \ldots$ *u* (A): $A_1, A_2, \ldots, A_n, \ldots$ *with the properties:*

1) The terms of the sequence (α) *are different one another and ordered so that* $0 < \alpha_k < \alpha_{k+1}$ *for* $k = 1, 2, 3, \ldots$*;*

2) The sequence (A) consists of nonzero numbers and it has finite numbers variations, i.e. there exists a natural number N, so that $A_k\, A_{k+1} > 0$ *for k>N;*

3) The functional sequence with the general term being the rational function

$$r_n(z) = \gamma + \sum_{k=1}^{n} \frac{A_k}{\left(z^2 - \alpha_k^2\right)}, \quad \gamma \in \mathrm{R},$$

is uniform convergent in every bounded domain which does not contain the points $\pm\alpha_k$ $(k = 1, 2, 3, \ldots)$.

Then the limit function $r(z) = \lim\limits_{n\to\infty} r_n(z)$ *has infinite number real zeros and not more than 2N+2 non-real ones. Moreover,* $r(z)$ *has finite number multiple zeros and, the zeros of* $r(z)$ *are separated by the points* $\pm\alpha_k$ *, from the certain place on.*

In L. Tchakalov [63] , an analogous statement is proved. The proof of Lemma 5.6.1 is carried out almost by the same way.

Let us denote, as it is commonly used, the zeros of $U_\nu(z)$ by

$$\pm j_{\nu,1}, \pm j_{\nu,2}, \ldots, \pm j_{\nu,k}, \ldots \qquad \left(0 < \pm j_{\nu,1} < \pm j_{\nu,2} < \ldots\right)$$

and let $\mu = \min(-1/2, \nu)$.

Lemma 5.6.2. [33] *Let* $f(t)$ *be a function, defined and bounded in the interval* [0, 1] *and let* $\int_0^1 |f(t)|\, t^\mu\, dt < \infty$. *Then the meromorphic function* $A_\nu(f;z)/U_\nu(z)$ *has the following representation*

$$\frac{A_\nu(f;z)}{U_\nu(z)} = \int_0^1 f(t)\, t^\nu\, dt - 2\sum_{k=1}^{\infty} \frac{A_\nu\left(f; j_{\nu,k}\right)}{j_{\nu,k}^2\, U_{\nu+1}\left(j_{\nu,k}\right)} \cdot \frac{z^2}{z^2 - j_{\nu,k}^2}. \qquad (5.6.1)$$

Moreover, the series on the right hand side of the above equation is uniformly convergent in every bounded domain which does not contain any one of the points $\pm j_{\nu,k}$ $(k = 1, 2, 3, \ldots)$.

Proof. Let us denote

$$R_\nu(z) = A_\nu(f;z)/U_\nu(z), \ \lambda = \nu\pi/2 + \pi/4 ,$$

and consider the contour integral

$$I_n(z) = \frac{1}{2\pi i}\int_{C_n}\left(\frac{1}{\zeta - z} - \frac{1}{\zeta}\right)R_\nu(\zeta)\,d\zeta, \quad n \in \mathbf{N}, \tag{5.6.2}$$

where C_n is a positively oriented rectangle with the vertices at the points $\pm(n + \nu/2 + 1/4)\pi \pm in$. We suppose that the complex number z is different from the poles of $R_\nu(\zeta)$ and that $n > |z| > 0$. Under these conditions there exists a positive integer N_1, such that for every $n > N_1$, the only singular points of the integrand inside in the contour C_n are the poles [28]:

$$\zeta = 0, \zeta = z, \zeta = \pm j_{\nu,k} \ (k = 1, 2, 3, \ldots, n).$$

The following residues correspond to them:

$$\mathrm{Res}\,0 = -\int_0^1 f(t)t^\nu\,dt, \qquad \mathrm{Res}\,j_{\nu,k} = \frac{A_\nu(f;j_{\nu,k})}{j_{\nu,k}^2\,U_{\nu+1}(j_{\nu,k})}\cdot\frac{z}{z - j_{\nu,k}},$$

$$\mathrm{Res}\,z = R_\nu(z), \qquad \mathrm{Res}(-j_{\nu,k}) = \frac{A_\nu(f;j_{\nu,k})}{j_{\nu,k}^2\,U_{\nu+1}(j_{\nu,k})}\cdot\frac{z}{z + j_{\nu,k}}.$$

By applying the residue theorem we get

$$I_n(z) = R_\nu(z) - \int_0^1 f(t) t^\nu \, dt + 2\sum_{k=1}^{n} \frac{A_\nu(f; j_{\nu,k})}{j_{\nu,k}^2 U_{\nu+1}(j_{\nu,k})} \cdot \frac{z^2}{z^2 - j_{\nu,k}^2}.$$

It can be proved that when the natural number n increases infinitely, then $I_n(z)$ vanishes. For this end, it is enough to show that there is a natural number N_2 and a constant M, so that $|R_\nu(z)| \le M|\zeta|^{1/2}$, when ζ remains on any contour C_n for $n > N_2$.

For estimating $|R_\nu(z)|$, we represent $R_\nu(z)$ in the form

$$R_\nu(z) = \frac{\int_0^{1/|\zeta|} f(t) t^\nu U_\nu(\zeta t) \, dt}{U_\nu(\zeta)} + \frac{\int_{1/|\zeta|}^{1} f(t) t^\nu U_\nu(\zeta t) \, dt}{U_\nu(\zeta)}. \qquad (5.6.3)$$

Using the asymptotic formula (2.1.4) for $\zeta \to \infty$, we can estimate any addend in (5.6.3).

First, let us consider $|R_\nu(z)|$ along the vertical sides of the rectangle

$$C_n:\ \zeta = \pm(n + \nu/2 + 1/4)\pi \pm i\eta, \quad -n \le \eta \le n.$$

Because of the evenness of $R_\nu(z)$ we can only consider the right hand vertical side. Let us denote

$$L_1 = \sup_{|\zeta|=1} |U_\nu(\zeta)|, \quad L_2 = \sup_{t\in[0,1]} |f(t)|.$$

We obtain consecutively

$$\frac{\left|\int\limits_0^{1/|\zeta|} f(t)t^{\nu}U_{\nu}(\zeta t)\,dt\right|}{|U_{\nu}(\zeta)|}=\frac{\left|\zeta^{\nu}\int\limits_0^{1/|\zeta|} f(t)t^{\nu}U_{\nu}(\zeta t)\,dt\right|}{|J_{\nu}(\zeta)|}$$

$$\le\frac{|\zeta|^{\nu+1/2}\int\limits_0^{1/|\zeta|}\left|f(t)t^{\nu}U_{\nu}(\zeta t)\right|dt}{\sqrt{2/\pi}\;\;\mathrm{ch}\,\eta\left|1-i\,\mathrm{th}\,\eta\,O(1/|\zeta|)\right|}.$$

Depending on the value of ν we can consider the following two cases:

1) $\nu>0$. Then $t^{\nu}\le|\zeta|^{-\nu}$ and therefore

$$\int\limits_0^{1/|\zeta|}\left|f(t)t^{\nu}U_{\nu}(\zeta t)\right|dt\le L_1L_2|\zeta|^{-\nu}\int\limits_0^{1/|\zeta|}dt=L_1L_2|\zeta|^{-\nu-1}.$$

2) $\nu\le 0$. Then $\nu+1/2\le 1/2$, from where we have the inequality $|\zeta|^{\nu+1/2}\le|\zeta|^{1/2}$. Further, letting $L_3=\int\limits_0^1|f(t)|t^{\nu}\,dt$, we get

$$\int\limits_0^{1/|\zeta|}\left|f(t)t^{\nu}U_{\nu}(\zeta t)\right|dt\le L_1\int\limits_0^{1/|\zeta|}|f(t)|t^{\nu}\,dt\le L_1L_3\,.$$

Let us note that $\mathrm{ch}\,\eta\ge 1$ and there exists a natural number N_3, such that $\left|1-i\,\mathrm{th}\,\eta\;O(1/|\zeta|)\right|\ge 1/\sqrt{2}$ for every $n>N_3$. If we denote

$$L=\begin{cases}L_1L_2\sqrt{\pi}, & for\ \nu>0\\ L_1L_3\sqrt{\pi}, & for\ \nu\le 0\end{cases},$$

we conclude that

$$\frac{\left|\int_0^{1/|\zeta|} f(t)t^{\nu}U_{\nu}(\zeta t)\,dt\right|}{|U_{\nu}(\zeta)|} \le L|\zeta|^{1/2} \qquad \text{for } n > N_3.$$

Now, let us estimate the second addend. We have

$$\frac{\int_{1/|\zeta|}^{1} f(t)t^{\nu}U_{\nu}(\zeta t)\,dt}{U_{\nu}(\zeta)} = \frac{\int_{1/|\zeta|}^{1} f(t)t^{-1/2}\left(\cos(\zeta t-\lambda)-\sin(\zeta t-\lambda)O(1)\right)dt}{(-1)^n \operatorname{ch}\eta\left(1-i\operatorname{th}\eta\, O(1/|\zeta|)\right)}.$$

Knowing that $|\cos(\zeta t-\lambda)| \le \operatorname{ch}\eta$ and $|\sin(\zeta t-\lambda)| \le \operatorname{ch}\eta$, we get that there exists a constant L_4 such that

$$\frac{\left|\int_{1/|\zeta|}^{1} f(t)t^{\nu}U_{\nu}(\zeta t)\,dt\right|}{|U_{\nu}(\zeta)|} \le L_4 \int_{1/|\zeta|}^{1} |f(t)|\, t^{-1/2}\, dt \quad \text{for } n > N_3.$$

In analogous way, for the horizontal side of the rectangle:

$$C_n:\ \zeta = \xi \pm i n, \quad -n\pi-\lambda \le \xi \le n\pi+\lambda,$$

There exist positive constants *P, Q* and a natural number N_4, such that for every $n > N_4$ the following inequalities hold true:

$$\frac{\left|\int\limits_0^{1/|\zeta|} f(t)t^{\nu}U_{\nu}(\zeta t)\,dt\right|}{|U_{\nu}(\zeta)|}\le P, \qquad \frac{\left|\int\limits_{1/|\zeta|}^{1} f(t)t^{\nu}U_{\nu}(\zeta t)\,dt\right|}{|U_{\nu}(\zeta)|}\le Q.$$

Let us denote $N_2 = \max(N_3, N_4)$ и $L_5 = L_4\int\limits_0^1 |f(t)|\, t^{-1/2}\, dt$. Let also $n > N_2$ and $M = 2\max(L, P, Q, L_5)$. Letting the point ζ located on the contour C_n it follows that $|R_{\nu}(z)| \le M|\zeta|^{1/2}$.

Finally, let $N = \max(N_1, N_2)$ and $n>N$. Bearing in mind (5.6.2), we get for $|I_n(z)|$ that

$$|I_n(z)| \le M\,\frac{4n+(2n+2\nu+1)\pi}{2n\pi\left(n-|z|\right)}\left((n\pi+\lambda)^2+n^2\right)^{1/4}|z| = O\!\left(n^{-1/2}\right).$$

The upper limit for $|I_n(z)|$ obtained above, vanishes when $n\to\infty$. Therefore

$$R_{\nu}(z) = \int\limits_0^1 f(t)t^{\nu}\,dt - 2\sum_{k=1}^{\infty}\frac{A_{\nu}(f;j_{\nu,k})}{j_{\nu,k}^2\,U_{\nu+1}(j_{\nu,k})}\cdot\frac{z^2}{z^2-j_{\nu,k}^2}.$$

The last to note is that the series on the right hand side of the equality above is uniformly convergent in every bounded domain which does not contain any of the points $\pm j_{\nu,k}$ $(k = 1, 2, 3, \ldots)$. □

5.7 Distribution of the Zeros

As it has been said in the begin of this chapter, under certain conditions of very general character on the function f, the entire function (5.5.1) has at most finite number non-real zeros. The zeros distribution is given by the theorem below.

Theorem 5.7.1. [33] *Let* $f(t)$ *be a real-valued function, defined and differentiable in the interval* [0, 1] *and let*

$$\int_0^1 |f(t)| t^{-3/2} dt < \infty, \quad \int_0^1 |f'(t)| t^{-1/2} dt < \infty, \quad and \quad f(1) \neq 0.$$

Then the function (5.5.1) *has at most finite number of non-real zeros and infinite number real ones. Besides,* (5.5.1) *has only finite number multiple zeros. From a certain place on, the zeros of* (5.5.1) *are separated by* $\pm j_{\nu,k}$ $(k = 1, 2, 3, \ldots)$.

Proof. Let $f(t)$ satisfies the conditions of the theorem. Let us denote:

$$C_{\nu,k} = \frac{A_\nu\left(f; j_{\nu,k}\right)}{U_{\nu+1}\left(j_{\nu,k}\right)}. \tag{5.7.1}$$

It can be proved that from a certain place on, the inequality $C_{\nu,k}\, C_{\nu,k+1} > 0$ holds true. To this end let us represent (5.7.1) in the form

$$C_{\nu,k} = \frac{\int_0^{1/j_{\nu,k}} f(t)t^{\nu} U_{\nu}\left(j_{\nu,k}\, t\right)dt}{U_{\nu+1}\left(j_{\nu,k}\right)} + \frac{\int_{1/j_{\nu,k}}^{1} f(t)t^{\nu} U_{\nu}\left(j_{\nu,k}\, t\right)dt}{U_{\nu+1}\left(j_{\nu,k}\right)},$$

and let us have in mind the asymptotic formula (2.1.4). After integration by parts and using the denotations:

$$s_{\nu,k} = f\left(1/j_{\nu,k}\right) J_{\nu+1}(1),$$

$$S_{\nu,k} = \int_{1/j_{\nu,k}}^{1} \left(f'(t)\, t^{-1/2} - f(t)\, t^{-3/2}\, O(1) \right) \sin\left(j_{\nu,k} t - \lambda\right) dt \ ,$$

We get

$$\frac{\int_0^{1/j_{\nu,k}} f(t)t^{\nu} U_{\nu}\left(j_{\nu,k}\, t\right)dt}{U_{\nu+1}\left(j_{\nu,k}\right)}$$

$$= \frac{s_{\nu,k} - \int_0^{1/j_{\nu,k}} J_{\nu+1}\left(j_{\nu,k}\, t\right)\left(f'(t) - (\nu+1)\frac{f(t)}{t}\right) dt}{\sqrt{\frac{1}{j_{\nu,k}}}\left(\sin\left(j_{\nu,k} - \lambda\right) + \cos\left(j_{\nu,k} - \lambda\right) O\left(\frac{1}{j_{\nu,k}}\right)\right)} \sqrt{\pi/2},$$

and

$$\frac{\int_{1/j_{\nu,k}}^{1} f(t)t^{\nu} U_{\nu}\left(j_{\nu,k}\, t\right)dt}{U_{\nu+1}\left(j_{\nu,k}\right)} = \frac{f(1)\sin\left(j_{\nu,k} - \lambda\right) - \sqrt{j_{\nu,k}}\; f\left(1/j_{\nu,k}\right) - S_{\nu,k}}{\sin\left(j_{\nu,k} - \lambda\right) + \cos\left(j_{\nu,k} - \lambda\right) O\left(1/j_{\nu,k}\right)}.$$

The functions $f'(t)t^{-1/2}$, $f(t)t^{-3/2}$ are integrable in the interval [0,1], so that $\lim_{t\to 0} f(t)t^{-1/2}=0$ and the function $\left|f'(t)-(\nu+1)f(t)/t\right|$ is a bounded one. Additionally

$$\lim_{k\to\infty} S_{\nu,k}=\lim_{k\to\infty}\int_{1/j_{\nu,k}}^{1}\left(f'(t)t^{-1/2}-f(t)t^{-3/2}O(1)\right)\sin\left(j_{\nu,k}t-\lambda\right)dt=0,$$

And since [28] the limit $\lim_{k\to\infty}\left(j_{\nu,k}-\left(k+\nu/2-1/4\right)\pi\right)=0$, then $\lim_{k\to\infty}\left|\sin\left(j_{\nu,k}-\left(\nu\pi/2+\pi/4\right)\right)\right|=1$. Therefore, there exists a natural number N, such that for every $k>N$ we have $\operatorname{sign} C_{\nu,k}=\operatorname{sign} f(1)$.

Let us consider again the equality (5.6.1) and let us denote

$$\gamma=\int_0^1 f(t)t^{\nu}dt-2\sum_{k=1}^{\infty}\frac{A_\nu\left(f;j_{\nu,k}\right)}{j_{\nu,k}^2\,U_{\nu+1}\left(j_{\nu,k}\right)}.$$

It is obvious that $|\gamma|<\infty$. We have

$$\frac{A_\nu(f;z)}{U_\nu(z)}=\gamma-2\sum_{k=1}^{\infty}\frac{C_{\nu,k}}{z^2-j_{\nu,k}^2}.$$

Therefore, due to Lemma 5.6.1, the function $A_\nu(f;z)/U_\nu(z)$ has not more $2N+2$ non-real zeros. The other details of the proof follow from Lemma 5.6.1. □

6 Mathematical Model of Non-Stationary Heat Convection of Power-Plant of Non-Piloted Flying Devices

6.1 Introduction

When solving a number of non-stationary problems of the heat convection theory, it is needed to find the dependence of the temperature field on the time at set geometrical sizes and physical characteristics of the body. The temperature field in a given initial moment of time is accepted as known.

At absence of a heat source, the equation of heat conductivity is described by the Fourier differential equation [25]:

$$\frac{\partial T}{\partial \tau} = a\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right). \tag{6.1.1}$$

When using an analythical method, the equation (6.1.1) is integrated at the corresponding conditions of uniqueness, but it is considerably complicated even for bodies with simple geometry, like a flat plate, a thick cylinder, etc., and the results are usually not convenient for practical use.

In relation to the solving of problems with finite limits of variation of variables, a developed method of finite integral transformations is given in [24].

In particular, when solving the problem of defining the non-stationary temperature fields for a hollow axially symmetrical cylinder in set boundary value conditions of third kind on the inner and outer surface, the following Hankel transformation is used (see Liykov [24])

$$H\left[T(r,z,\tau)\right]=\int_{R_1}^{R_2} T(r,z,\tau)\,rU_0(\mu_m\, r/R_1)\,dr,$$

where U_0 is a linear combination of Bessel functions of first and second kind, and μ_m – roots of the equation

$$\frac{U_0(\chi\mu)}{U_1(\chi\mu)}=\frac{\chi\mu}{Bi_2} \quad (m=1,\ 2,\ \ldots).$$

Comparing the mathematical methods of evaluations of the temperature fields for bodies with a different geometrical shape, the purpose of the current work is to make a mathematical model of the non-stationary temperature fields of power-plant combustion chambers in non-piloted flying devices, working in the conditions of strong throttling.

6.2 Mathematical modeling of non-stationary heat convection of a power-plant with a liquid-rocket engine

Transitional work rates of liquid-rocket engines (LiRE), i.e. mainly the rates of starting and stopping, represent interconnected non-stationary processes in the gas-generating system, the combustion chamber and the turbo-pump aggregate, which impose a series of restrictions, serving the safety and reliability of the power-plant as a whole.

Along these lines, in the literature different methods - analytic, numeral and mixed, are described. A comparison about their precision and duration of accounts is made in Delft University of Technology - The Netherlands [62]. Four theoretical models are compared: analytical, numeric-analytical, approximately analytical, and numerical integration. The engine chamber is divided into separate circle segments, in each of which the heat stream is radially-only distributed and in the initial moment of time the temperature in the volume is constant. The precision of results obtained and the loss of computing time are accepted as criteria of effectiveness. The approximate analytical model is the most effective if the ratio of the thickness of the chamber wall to the radius is small enough. The analytical method based on integration of the Fourier heat transfer differential equation under the given conditions of uniqueness has proved to be the most precise for the comparison. But it is too complicated, especially for bodies with composite geometry or more complex initial and boundary conditions.

Also, a purely analytical solution of the so-called initial heat amplitudes A_i containing a combination of the Bessel first kind functions is found, and the precision is great even for high values of the time τ, for reversed value of the combustion chamber wall thickness about 0,5. The precision of the rest methods is great enough, so it is not considered a determinant for the choice of a certain method [62].

The non-stationary temperature fields (in a LiRE combustion chamber with a small pulling power) working at an impulsive rate are researched in the so-called "post-effective" period, when giving out a command for ending the load of combustion components described in the paper [48].

In the current work a mathematical model is made of the non-stationary heat convection in combustion chambers of power plants of non-piloted flying devices permitting to predict the heat condition and to increase their reliability.

When throttling LiRE-s, which are not made to work at such rates, it is very important to determine the non-stationary temperature fields and the combustion chamber wall temperature (from the cooling component side) at certain conditions and type of the power-plant. At such work rates the cooling component considerably decreases which creates a danger of overheating the combustion chamber.

Besides, the combustion chamber with the inner and outer surface radiuses, respectively R_1 and R_2 in an initial moment of time τ_0, has one and the same temperature T_0. At a moment of starting the engine it is heated by the powerful convection a ray stream of the hot gases for which it is accepted to have a constant temperature T_g.

Let us make the following suggestions:

- the combustion chamber is made of a homogenous isotropic material;

- the volume expansion caused by the change of temperature is accepted as little enough to be ignored;
- the thermal physical properties of the wall material, and particularly the heat convection coefficient λ_w, density ρ_w, thermal capacity c_w, in the working temperature field are constant;
- the curve of the combustion chamber in the area of the critical cross-section and the nozzle is disregarded;
- there are set conditions of third kind on the inner and outer combustion chamber wall.

The generated heat transfer on both chamber sides is different and does not vary in axial direction. The temperature field $T(\tau)$ in a non-stationary case is described by the heat transfer differential equation taken in cylindrical coordinates, and in particular

$$\frac{\partial T}{\partial \tau} = a\left(\frac{\partial^2 T}{\partial \bar{r}^2} + \frac{1}{\bar{r}}.\frac{\partial T}{\partial \bar{r}}\right). \tag{6.2.1}$$

In a dimensionless form it looks like

$$\frac{\partial \theta(\bar{r},Fo)}{\partial Fo} = \frac{\partial^2 \theta(\bar{r},Fo)}{\partial \bar{r}^2} + \frac{1}{\bar{r}^2}\frac{\partial \theta(\bar{r},Fo)}{\partial \bar{r}}, \tag{6.2.2}$$

where T is the cylindrical chamber temperature at a distance r from the axis, in a moment of time τ;

$\bar{r} = r/R_2$ - relative radius;

$\theta(\tau) = \dfrac{T - T_0}{T_g - T_0} = \Phi\left(\dfrac{r}{R}, Fo, Nu\right)$ - dimensionless cylindrical chamber temperature;

$Fo = \frac{a\tau}{R_2^{\ 2}}$ - Fourier criterion;

a– temperature transfer coefficient;

τ- time;

R_2 – outer chamber surface radius;

$Nu_{1,2} = \frac{\alpha R_{1,2}}{\lambda_w}$ - Nusselt criteria on the respective chamber wall side (the cooler);

α - temperature conductivity coefficient;

λ_w – heat transfer coefficient of the chamber constructing metal;

The uniqueness conditions, which give a full mathematical description of the process together with (6.2.1):

- geometrical conditions: $M \le \bar{r} \le 1;$
- physical conditions: $\alpha = const, \ a = const, \ \lambda_w = const;$
- initial conditions: $\theta(\bar{r},0) = 0;$
- boundary conditions:

$$\frac{\partial\theta(M,Fo)}{\partial\bar{r}} = Nu_1\left[\theta(M,Fo) - 1\right]_{\bar{r}=M};$$

$$\frac{\partial\theta(1,Fo)}{\partial\bar{r}} = -Nu_2\left[\theta(1,Fo) - \theta_S\right]_{\bar{r}=1}.$$

The Nusselt criteria on the boundary “hot gases – inner chamber side” and respectively on the boundary “outer chamber side – cooler” are:

$$Nu_1 = \frac{\alpha_g . R_1}{\lambda_w}, \qquad Nu_2 = \frac{\alpha_f . R_2}{\lambda_w}.$$

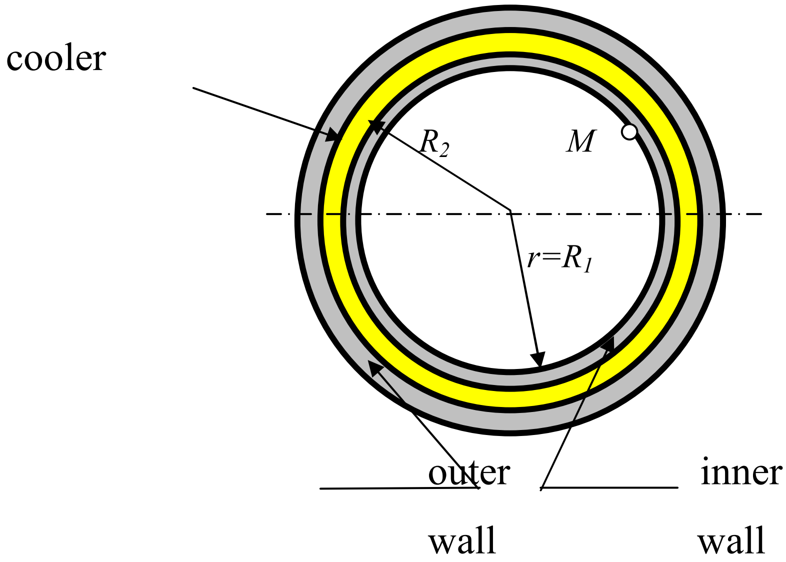

Figure 6.2.1. A cross-section of a liquid-rocket combustion chamber

The idea is to use a Hankel finite integral transformation for a hollow axially symmetrical cylinder with boundary conditions of the third kind on the inner and outer combustion chamber side in the form (*Liykov* [24]) with a kernel $V_0(\delta\ \bar{r})$:

$$\bar{\theta}(\delta, Fo) = H[\theta(\bar{r}, Fo)] = \int_M^1 \bar{r}.\theta(\bar{r}, Fo).V_0(\delta\bar{r})d\bar{r},$$

$$V_0(\delta\bar{r}) = [\delta Y_1(M\delta) + Nu_1 Y_0(M\delta)]I_0(\delta\bar{r}) - [\delta.I_1(M\delta) + Nu_1 I_0(M\delta)]Y_0(\delta\bar{r}),$$

where $I_0(\delta\ \bar{r})$ and $I_1(\delta\ \bar{r})$ are the modified Bessel functions of the first kind, and $Y_0\ (\delta\ \bar{r})$ and $Y_1\ (\delta\ \bar{r})$ – the Bessel functions of the second kind.

If in addition to this $V_1(\delta\ \bar{r})$ is the linear combination of Bessel functions of the first and the second kind:

$$V_1(\delta\bar{r}) = [\delta Y_1(M\delta) + Nu_1 Y_0(M\delta)] I_1(\delta\bar{r}) - [\delta I_1(M\delta) + Nu_1 I_0(M\delta)] Y_1(\delta\bar{r}),$$

then δ represents the positive roots of the characteristic equation

$$\delta V_1(\delta) - Nu_2 V_0(\delta) = 0 .$$

Such a solution is used for predicting the heat condition of not refrigerate-liable power-plant combustion chambers with small pulling power and impulsive action (Prysnyakov, Serebryansky [48]).

In this case, we use the Hankel transformation for evaluation of the non-stationary chamber temperature fields in a strongly throttled engine with outer regenerative cooling in a moment of starting.

Applying the Hankel transformation and the respective boundary conditions to the heat transfer equation and after integration with respect to the variable subject to exclusion, as a result instead of a partial differential equation for the original of the function, an ordinary first order differential equation for the image $\bar{\theta}(\delta, Fo)$ of the function $\theta(\bar{r}, Fo)$ is obtained, and namely

$$A\frac{d\bar{\theta}}{dFo} + \bar{\theta} = Kq_{\Sigma};$$
$$\bar{\theta}(0) = 0;$$

$$A = \frac{1}{\delta^2}, \qquad K = \frac{1}{\delta^2}\left[MNu_1 V_0(M\delta) + Nu_2 \theta_c V_0(\delta)\right]. \qquad (6.2.3)$$

Very important in the further explanation are the positive roots δ_i ($i = 1, 2, 3,....$) of the charecteristic equation

$$\frac{V_0(\delta)}{V_1(\delta)} = \frac{1}{Nu_2}\delta . \qquad (6.2.4)$$

The total heat stream q_Σ is a single entry signal of an inertion system, whose dynnamical qualities are described by equation (6.2.2). We accept the system is subjected to the action of a rectangular impulse with an amplitude K and a period Fo. The reaction is an exponent which after returning to the original is [48]:

$$\theta(\bar{r}, Fo) = \sum_{i=1}^{\infty} \frac{V_0(\delta_i \bar{r})\bar{\theta}(\delta_i, Fo)}{\left[1+\left(\frac{Nu_2}{\delta_i}\right)^2\right] V_0(\delta_i) - \frac{4}{\pi^2}\left[1+\left(\frac{Nu_1}{\delta_i}\right)^2\right]}; , \qquad (6.2.5)$$

$$\bar{\theta}(\delta_i, Fo) = K[1-\exp(-Fo/A)]. \qquad (6.2.6)$$

The right side of equation (6.2.4) is an equation of a straight line of the type $y=kx$. When the criterion $Nu_2 \to \infty$ (practically for $Nu_2>100$), $k \to 0$, $\alpha \to 0$, i.e. the straight line matches with the abscissa, the characteristic equation roots are determined by

$$V_0(\delta) = 0 .$$

For a given value of M on the system Maple V for Windows, the characteristic equation roots for different values of the Nusselt criteria on the outer chamber wall can be found, i.e. in dependence on the heat stream, the geometry and the coefficient of the wall heat conductivity.

For this purpose, it is useful to localize first the positive roots of the charecteristic equation (6.2.4). Thus it is enough (*fig.2*) to represent graphically the function $\frac{V_0(\delta)}{V_1(\delta)} - \frac{\delta}{Nu_2}$ for several values of Nu_1 and Nu_2.

A more detailed view (*fig. 3*) shows that for all considered values of Nu_1 and Nu_2, the roots of the charecteristic equation (6.2.4) are "close" to those in *fig.2*.

After that the first 5 charecteristic equation positive roots for $Nu_1 = 2\kappa$ $(k = 1 \div 50)$ and $Nu_2 = 100l$ $(\, l = 1 \div 6,\ l = 1, 2, 3, 4, 5, \ldots, 15)$ are tabulated. Also, the first 5 positive roots of the equation $V_0(\delta)=0$ are found. The analysis of the obtained results confirms the suggested conclusion, that with the increasing of Nu_2, the tabulated roots of the charecteristic equation (6.2.4) approximate to the roots of the equation $V_0(\delta)=0$.

Let us substitute (6.2.3) and (6.2.6) in the equality (6.2.5) and denote

$$W_i(\bar{r},\delta_i) = \frac{V_0(\delta_i \bar{r})}{\left[1+\left(\frac{Nu_2}{\delta_i}\right)^2\right].V_0(\delta_i) - \frac{4}{\pi^2}\left[1+\left(\frac{Nu_1}{\delta_i}\right)^2\right]}.$$

Then, the equality (6.2.5) for the sought dimensionless temperature can be represented in the form

$$\theta(\bar{r},Fo) = \sum_{i=1}^{\infty} W_i(\bar{r}\delta_i)K(1-exp(-Fo/A_i)), \text{ where } A_i = \frac{1}{\delta_i^2}. \qquad (6.2.7)$$

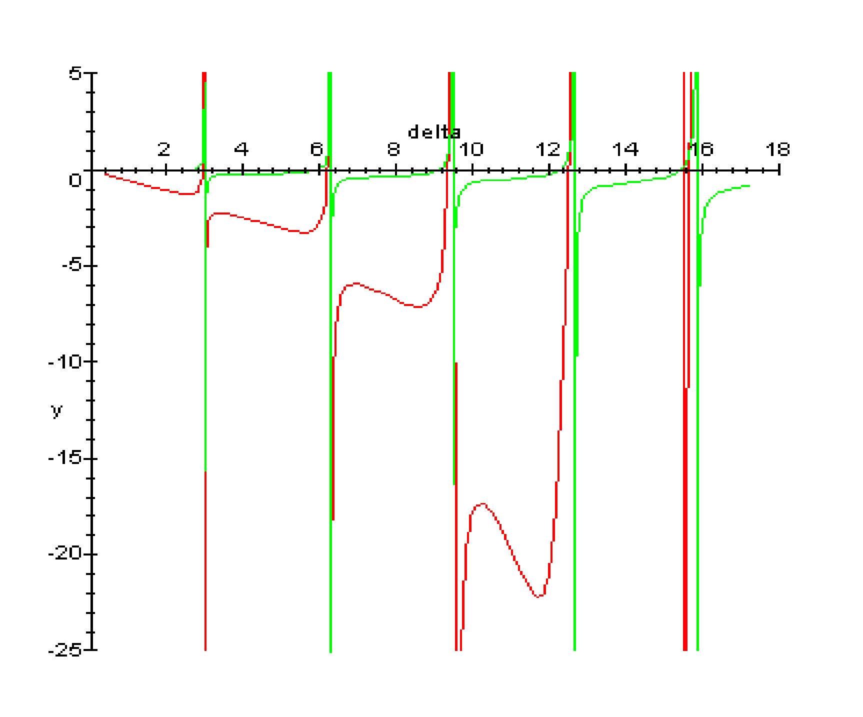

Figure 6.2.2. Localization of the first 5 positive roots of equation (6.2.4) for $Nu_1 = 2$, $Nu_1 = 38$, $Nu_2 = 100$ in a diapazone from 0 to + 18 (values on the axe y from –25 to +5)

Therefore the problem is to find the function $W_i(\bar{r},\delta_i)$ in dependence on the heat transfer conditions of the two chamber wall sides. These values can be tabulated or graphically represented, which allows us to define the function $W_i(\bar{r},\delta_i)$ for different heat streams and coolers, used for given liquid-rocket engines.

Note that the distribution of the function $W_i(\bar{r})$ for different gas stream temperatures in the combustion chamber and the nozzle, and different Nousselt criteria on the inner and outer surface is graphically shown in fig. 6.3.1.

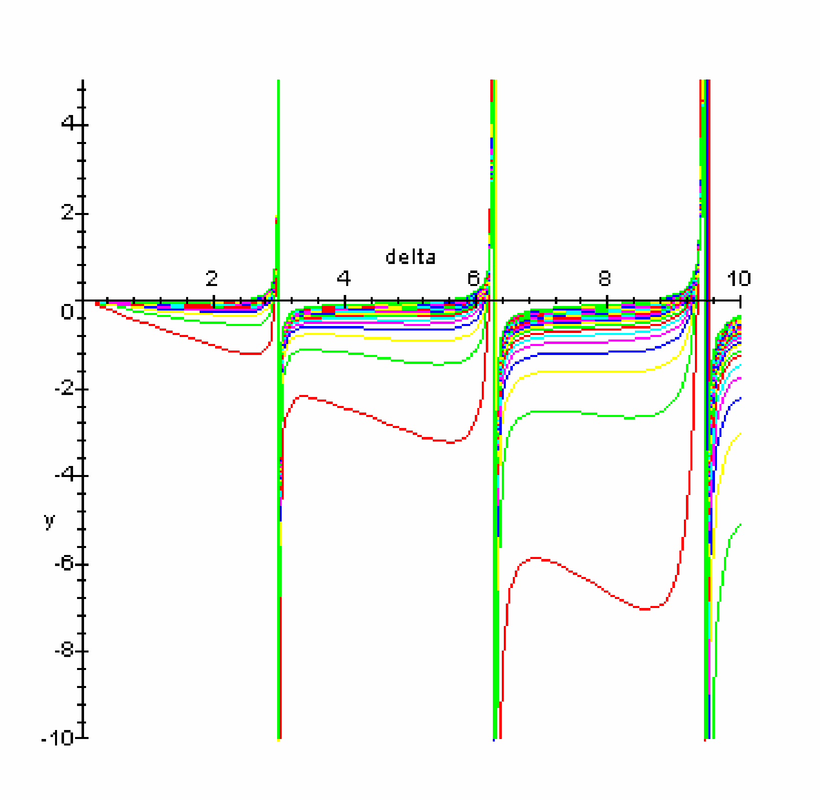

Figure 6.2.3. Localization of the roots of equation (6.2.4) for $Nu_1 = 2 - 100$ and $Nu_2 = 1500$ in a diapazone form 0 to + 10

6.3 A discussion of the obtained results

The intensity of heating in a liquid-rocket engine combustion chamber in a moment of starting is characterized by the speed of the relative temperature increasing, defined by the formula

$$\theta(\tau) = \frac{T - T_0}{T_g - T_0}, \qquad (6.3.1)$$

where $T(\tau)$ is the current value of the temperature in a moment of τ; T_0 – the initial temperature of the elements in the chamber construction; T_g – the temperature of the medium (the gas steam).

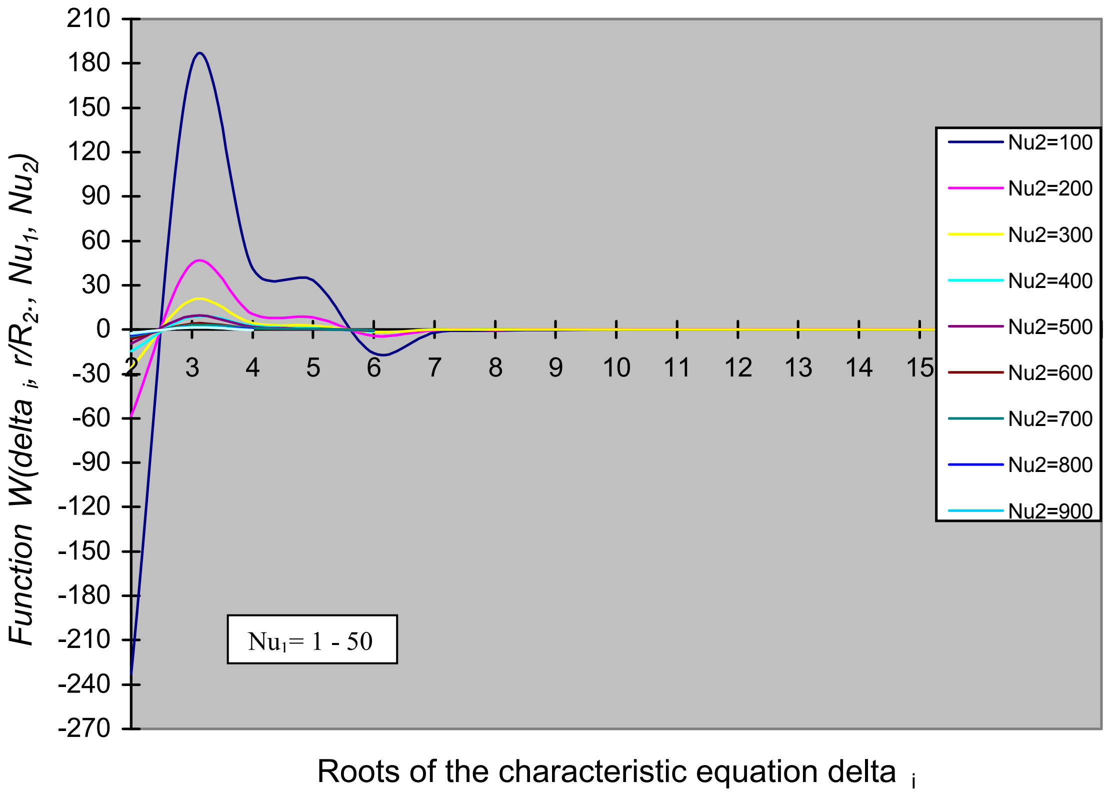

Figure 6.3.1. Graphical representation of the function $W_i(\bar{r},\delta_i)$ for roots of the characteristic equation $\delta_i \in (2,16)$ and the Nusselt criteria, $Nu_1 = 1 – 50$ and $Nu_2 = 100 – 1000$

In the current Chapter is done a modeling of non-stationary temperature fields in the liquid-rocket engine chambers with a regenerative cooling of non-piloted flying devices, based on the use of the Hankel transformation for solving the heat conductivity differential equation for the respective boundary conditions and uniqueness.

The roots of the characteristic equation, whose solving is done on the CAS Maple 13 for Windows, are tabulated in dependence on different values of the criterion Nu_1. The dependence of δ_i on the criterion Nu_2 is considerably weak and for values of Nu_2 larger than 100, δ_i does not depend on it almost at all.

Tabular and graphical dependences are created, such that allow to define the functions $W_i(\bar{r}\delta_i)$ for given conditions of the heat conductivity on both sides of the chamber walls and geometry of different liquid-rocket engines.

With the help of the functions $W_i(\bar{r})$ can be found the distribution of $\theta(\tau)$ by formula (6.3.1) in dependence on the roots of the characteristic equation δ_I and the Fourier criterion $Fo = \dfrac{a\tau}{R_2^{\ 2}}$,

where a is a heat transfer coefficient of the chamber material [w/m^2K];
τ – the current time [sek];
R_2 – the chamber diameter or the nozzle from the side of the cooling liquid [m].

Thus, we can prognosticate about the intensity of heating in the chamber or the nozzle of liquid-rocket engines in starting rates and reaching an almost “stationary” rate.

The modeling by using the Hankel transformation allows suggesting about the heat condition of the combustion chamber construction elements in power-plants, working at rates that differ from the standard ones, and to find preliminarily some singularities of their heat load at starting rate, considerably influencing the strain-deformational condition and the reliability of the power-plant as a whole.

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